Lecture 27

Monday, March 31, 2025 1:13 PM

Heat equation on a rod: $u_t = cu_{xx}$

This is a partial differential equation (PDE) rather than ordinary differential equation (ODE) because the unknown u is a function of more than one variable. Here, $u = u(x, t)$ is the temperature at position x on the rod at time t, and c is the *thermal diffusivity coefficient*.

Suppose the rod is the interval $[0,1]$.
Initial condition: $u(x, 0) = u_0(x)$
Boundary conditions: $u(0, t) = f_1(t)$ and $u(1, t) = f_2(t)$.

The initial condition and boundary conditions are typically controllable or measurable. So, u_0, f_1, f_2 can be considered as something given rather than something to be found.

Discretize the rod: $0 = x_0 < x_1 < \cdots < x_n = 1$ with *spatial step size* $h = x_i - x_{i-1}$
Discretize the time: $0 < t_1 < t_2 < \cdots$ with *time step size* $\tau = t_j - t_{j-1}$

The partial differential evaluated at (x_i, t_j):

$$u_t(x_i, t_j) = cu_{xx}(x_i, t_j) \qquad (1)$$

Let $u_{i,j} = u(x_i, t_j)$. Approximate the derivatives using finite difference method:

$$u_t(x_i, t_j) \approx \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} = \frac{u_{i,j+1} - u_{i,j}}{\tau}$$

$$u_{xx}(x_i, t_j) \approx \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

Equation (1) becomes:

$$\frac{u_{i,j+1} - u_{i,j}}{\tau} = \frac{c}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

which is equivalent to

$$u_{i,j+1} = u_{i,j} + a\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

where $a = c\tau/h^2$. The number a is called *Courant number* of the recursive formula. For the recursive formula to give reasonable results, it is required that $a < 1/2$.

Example:
With $c = 0.5$, $u_0(x) = \sin(\pi x)$, $f_1(t) = f_2(t) = 0$, find the temperature at position $x = 0, 0.25, 0.5, 0.75, 1$ at time $t = 0.1$. Use spatial step size $h = 0.25$ and time step size $\tau = 0.05$.