Lecture 3

Friday, January 17, 2025 12:50 PM

The process of transforming a matrix into an RREF is called *row reduction* or *Gauss-Jordan elimination*.
<u>Goal for today</u>: solve a system of linear equations using Gauss-Jordan elimination.

Example 1: consider a system of linear equations:

$$\begin{cases} x + 2y + 3z = 5 \\ 2x - y - z = -1 \\ y + z = 1 \end{cases}$$

Associated matrix (augmented matrix):

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Observation: any elementary row operation on this matrix results in an equivalent system of linear equation. Consequently, the RREF of the above matrix corresponds to an equivalent system of linear equation. This new system is very easy to solve.

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This new matrix (RREF) corresponds to the system

$$\begin{cases} x = 0 \\ y = -2 \\ z = 3 \end{cases}$$

Example 2: consider the system

$$\begin{cases} x - y + z = 1 \\ 2x + y = 2 \\ 4x - y + 2z = 3 \end{cases}$$

Associated matrix:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 4 & -1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{on the way to RREF}} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The new matrix corresponds to a system whose last row is $0x + 0y + 0z = -1$. This system is inconsistent (i.e. having no solutions). So is the original system.

Conclusion: if you can row-reduce an associated matrix into a matrix that has a row 0 0 0 … 0 a where $a \neq 0$ then the linear system has no solutions.

Example 3: consider a system of linear equations:

$$\begin{cases} x - y + z = 0 \\ 2x + y - z = 1 \\ 4x - y + z = 1 \end{cases}$$

Associated matrix (augmented matrix):

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & -1 & 1 \\ 4 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix corresponds to the system

$$\begin{cases} x = \frac{1}{3} \\ y - z = \frac{1}{3} \end{cases}$$

The third column of the matrix is doesn't have a pivot 1. The variable corresponding to this column, which is z, can be chosen as a "free" variable (meaning it can take an arbitrary value). Therefore,

$$z = t, y = t + \frac{1}{3}, x = \frac{1}{3}$$

One can rearrange as $(x, y, z) = \left(\frac{1}{3}, t + \frac{1}{3}, t\right)$, where $t \in R$.