Lecture 4

Wednesday, January 22, 2025 9:37 AM

The goal of the next 3 lectures is to help you understand and solve linear programming (also called *linear optimization*) problems. A *linear program* is a method to obtain the best outcome (minimum or maximum) of a linear function under some linear constraints. There are many available linear programs, but we will focus on the *Simplex method*. It is the simplest algorithm and can be performed by hand.

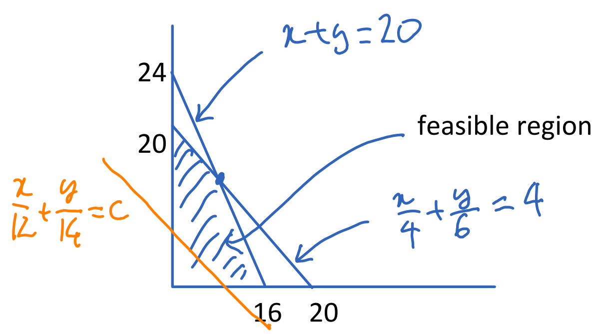
Motivating example: You work two part-time jobs, as a Math Tutor ($14/hour) and at PCC ($12/hour). Each hour working as a Math Tutor requires 15 min of prep time. Each hour working at PCC requires only 10 minutes of prep time. You are allowed to work at most 20 hours a week. Due to school work, you don't want to spend more than 4 hours a week for preparation. Find the number of working hours for each job per week to maximize your income.

Let x be the number of hours working as Math Tutor, and y be the number of hours working at PCC. Constraints:

$$x, y \geq 0$$
$$x + y \leq 20$$
$$\frac{x}{4} + \frac{y}{6} \leq 4$$

Objective function: $f(x, y) = 14x + 12y$
This is the function you want to maximize.

The problem can be solved *graphically*.



Keep in mind that the line $\frac{x}{a} + \frac{y}{b} = 1$ is the line that intersects the x-axis at $(a, 0)$ and the y-axis at $(0, b)$. The line $14x + 12y = c$ is parallel to the line $14x + 12y = 14 \cdot 12$, which is equivalent to

$$\frac{x}{12} + \frac{y}{14} = 1$$

Feasible region is the set of all pairs (x, y) that satisfy the constraints of the problem. It is a region bounded by lines.

Objective function is the function we want to maximize or minimize. If it is to be minimized, it is often called a *cost function*.

In the previous problem, we need the find a point in the feasible region at which c is maximum. This can occur only at one of the vertices (i.e. corner points) of the feasible region. There are four corner points: $(0,0)$, $(16,0)$, $(0,20)$, (a, b) where (a, b) are coordinates of the intersection point of the lines

$$\mathrm{x} + \mathrm{y} = 20$$
$$\frac{\mathrm{x}}{4} + \frac{y}{6} = 4$$

We solve this system of equations using the Gauss-Jordan elimination method:

$$\begin{bmatrix} 1 & 1 & 20 \\ 1/4 & 1/6 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 12 \end{bmatrix}$$

Which yields $(a, b) = (8,12)$. Compare the objective function $f(x, y)$ at the four corner points gives the maximum at $(x, y) = (8, 12)$ with $f(8,12) = 256$.

A *general linear programing problem* is of the form of maximizing or minimizing the objective function

$$f(x_1, x_2, \dots, x_n) = a_1x_1 + a_2x_2 + \dots + a_nx_n$$

under the constraints

$$a_{11}x_1 + \dots + a_{1n}x_n \le (\ge) b_1$$
$$a_{21}x_1 + \dots + a_{2n}x_n \le (\ge) b_2$$
$$\dots$$
$$a_{k1}x_1 + \dots + a_{kn}x_n \le (\ge) b_k$$

Comment: the graphic method is intuitive and simple. However, it is only applicable for a simple problem where there are two, or maybe three, variables. But even with two or three variables, it can get complicated if there are a lot of constraints.

We will discuss an alternative method that can deal (still by hand) with a larger linear optimization problem. It is called the *simplex method*.

$x, y \ge 0$
$x + y \le 20$ (1)
$\frac{x}{4} + \frac{y}{6} \le 4$ (2)

Objective function: $f(x, y) = 14x + 12y$

Step 1: turn the problem into a *standard form*. This is a form in which
- must be a maximization problem,
- all linear constraints must be in a less-than-or-equal-to inequality,
- all variables are non-negative

The example under consideration is already in standard form.

Step 2: introduce slack variables to turn all inequality constrains to equality constraints:

$x, y, s_1, s_2 \ge 0$
$x + y + s_1 = 20$ (3)
$\frac{x}{4} + \frac{y}{6} + s_2 = 4$ (4)
If the right hand side of any equation is negative, multiply that equation by -1.
We want to maximize $z = 14x + 12y$.

Observation: consider the linear system of 3 equations and 5 unknowns

$$x + y + s_1 = 20$$
$$\frac{x}{4} + \frac{y}{6} + s_2 = 4$$
$$-14x - 12y + P = 0$$

This system is associated with the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ 1/4 & 1/6 & 0 & 1 & 0 & 4 \\ -14 & -12 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Recall that any elementary row operation transforms this matrix to a matrix that corresponds to an equivalent system of equations. So, what is the matrix corresponding to the simplest linear programming problem? Take a look at two following examples:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 5 \\ 0 & 2 & 1 & 0 & 1 & 7 \end{bmatrix}$$

The last row reads $2x_2 + s_1 + P = 7$ which implies $P = 7 - 2x_2 - s_1$. Then max $P = 7$, attained when $x_2 = s_1 = 0$. The first row implies $x_1 = 2$. The solution $(x_1, x_2) = (2,0)$ is acceptable (i.e. in the feasible region).

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 & 0 & 5 \\ 0 & 2 & 1 & 0 & 1 & 7 \end{bmatrix}$$

The last row reads $2x_2 + s_1 + P = 7$ which implies $P = 7 - 2x_2 - s_1$. Then max $P = 7$, attained when $x_2 = s_1 = 0$. The first row implies $x_1 = -2$. However, the solution $(x_1, x_2) = (-2,0)$ is not acceptable (i.e. outside of the feasible region).

Therefore, matrix A is more favorable situation.

Our goal in the next step (Step 3) is to transform this matrix into a matrix where

- all coefficients on the left of the column of P on the last row are nonnegative,
- all coefficients on the right column and above the last row are nonnegative.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ 1/4 & 1/6 & 0 & 1 & 0 & 4 \\ -14 & -12 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{want} \geq 0$$

want ≥ 0

Step 3:

- Locate the key column. This is the column containing the largest negative coefficient of the last row.
- Locate the pivot element on the key column. This is the element $a_j > 0$ such that the quotient b_j/a_j, where b_j is the far right coefficient on row j, is smallest. This guarantees that the coefficients on the right hand side is always nonnegative.
- Use elementary row operations to turn the key column into a column in which that pivot element becomes 1 and the rest become 0s.
- Repeat the first 3 sub-steps above with the new matrix. Stop when all coefficients on the left of the column of P on the bottom row are nonnegative.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ 1/4 & 1/6 & 0 & 1 & 0 & 4 \\ -14 & -12 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_3=R_3+14R_2]{\substack{R_2=4R_2 \\ R_1=R_1-R_2}} \begin{bmatrix} 0 & 1/3 & 1 & -4 & 0 & 4 \\ 1 & 2/3 & 0 & 4 & 0 & 16 \\ 0 & -8/3 & 0 & 56 & 1 & 224 \end{bmatrix}$$

$$\xrightarrow[R_3=R_3+(8/3)R_1]{\substack{R_1=3R_1 \\ R_2=R_2-(2/3)R_1}} \begin{bmatrix} 0 & 1 & 3 & -12 & 0 & 12 \\ 1 & 0 & -2 & 12 & 0 & 8 \\ 0 & 0 & 8 & 24 & 1 & 256 \end{bmatrix}$$

This matrix corresponds to the system:
$y + 3s_1 - 12s_2 = 12$
$x - 2s_1 + 12s_2 = 8$
$8s_1 + 24s_2 + P = 256$

Note that the last equation can be rewritten as
$P = 256 - 8s_1 - 24s_2$
Obviously, $\max P = 256$, attained when $s_1 = s_2 = 0$.

Example:
Maximize $3x_1 + 2x_2 + 5x_3$ under the constraints

$x_1 + 2x_2 + x_3 \geq 43$
$3x_1 + 2x_3 \leq 46$
$x_1 + 4x_2 \leq 42$
$x_1, x_2, x_3 \geq 0$