Lecture 1

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Consider some equations:
1) $3x + 7 = 0$. This is easy to solve: $x = -7/3$.
2) $3x^2 + 7x - 1 = 0$. This is also easy, although more complicated: $x = \frac{-7 \pm \sqrt{(-7)^2 - 4(3)(-1)}}{6}$.
3) $3x^3 + 7x - 1 = 0$. This is a cubic equation. The solution is given by a highly complicated formula, called Cardano's formula (1545).

Even if an exact form of the solution is available, in practice one only uses an approximate value. One reason is because of rounding off. The question is, given a certain threshold of error, can one find an approximate solution within that threshold?

Let's consider an example: find all real solutions to $x^3 - 3x + 1 = 0$.
You would try to factor the cubic polynomial $f(x) = x^3 - 3x + 1$. However, it doesn't have a rational root. You know that it has at most 3 roots because f is a polynomial of degree 3.

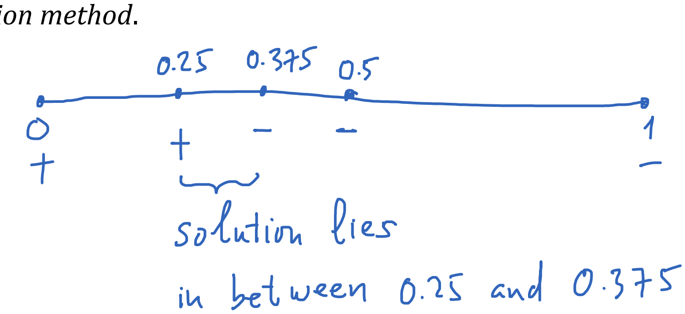
Recall the **Intermediate Value Theorem**: (you learned in Calculus I)
Let f be a continuous function on $[a,b]$ and let m be a number between $f(a)$ and $f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = m$.

Notice that
$f(-2) = -1 < 0$,
$f(-1) = 1 > 0$,
$f(0) = 1 > 0$,
$f(1) = -1 < 0$,
$f(2) = 3 > 0$.
The Intermediate Value Theorem implies that:

- There exists $c_1 \in (-2,-1)$ such that $f(c_1) = 0$.
- There exists $c_2 \in (0,1)$ such that $f(c_2) = 0$.
- There exists $c_3 \in (1,2)$ such that $f(c_3) = 0$.

Thus, we know that f has *exactly* three roots: one between -2 and -1, one between 0 and 1, and one between 1 and 2.

How do we find, say, c_2 ? There is a very simple numerical method for this purpose, called *bisection method*.



Bisection method:
Step 1: find an interval $[a,b]$ such that $f(a)$ and $f(b)$ have different signs.
Step 2: if $f\left(\frac{a+b}{2}\right)$ and $f(a)$ have different signs then narrow the search interval to $\left[a, \frac{a+b}{2}\right]$. Otherwise, narrow the search interval to $\left[\frac{a+b}{2}, b\right]$.
Keep going until the search interval is thin enough.