

Lecture 10: Interpolation Continuation (02/02/2026)

0.6 Newton's Divided Difference Method

Newton's Divided Difference method is a way to build a polynomial that passes through given data points step by step. It uses calculated slopes, called divided difference to determine the coefficients of the polynomial. Each new term is added gradually using factors like $(x - x_0)$ to determine the coefficients of the polynomial which allows the polynomial to adjust and include one additional data points at a time.

$$(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$$

$$P(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \dots + c_n(x - x_0) \dots (x - x_{n-1})$$

$$c_k = \underbrace{f[x_0, x_1, \dots, x_k]}_{\text{k'th divided difference}}$$

$$c_0 = f[x_0] = f(x_0) = y_0$$

$$f[x_0, x_1, \dots, x_k] = \frac{f[x_1, x_2, \dots, x_k] - f[x_0, x_1, \dots, x_{k-1}]}{x_k - x_0}$$

$$c_1 = f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$P(x_1) = c_0 + c_1(x_1 - x_0)$$

$$c_1 = \frac{P(x_1) - c_0}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

Example

$$(-1, 2), (0, 4), (1, 2), (2, 3)$$

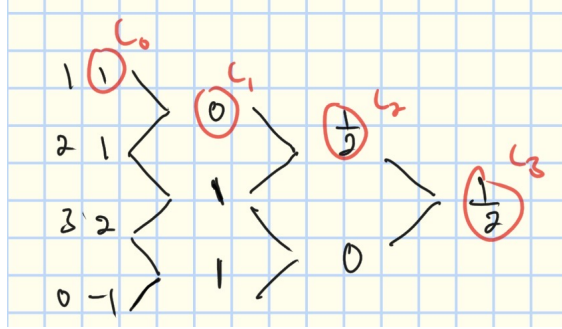


Figure 3: Newton Divided Difference

$$\begin{aligned}
 P(x) &= c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) \\
 &= 1 + 0(x - 1) + \frac{1}{2}(x - 1)(x - 2) + \frac{1}{2}(x - 1)(x - 2)(x - 3) \\
 &= 1 + \frac{1}{2}(x^2 - 3x + 2) + \frac{1}{2}(x^3 - 6x^2 + 11x - 6) \\
 &= 1 + \frac{1}{2}x^2 - \frac{3}{2}x + 1 + \frac{1}{2}x^3 - 3x^2 + \frac{11}{2}x - 3 \\
 &= \frac{1}{2}x^3 - \frac{5}{2}x^2 + 4x - 1
 \end{aligned}$$

0.7 Error Estimate

$$f(x) \approx P(x)$$

f and P match at $(x_0, y_0), \dots, (x_n, y_n)$

f is a non-polynomial function.

We want to estimate the error:

$$\begin{aligned}
 \max |f(x) - P(x)| \\
 x \in [x_0, x_n]
 \end{aligned}$$

Theorem:

Suppose $a = x_0 < x_1 < x_2 < \dots < x_n = b$

For each $x \in (a, b)$, there exists $r_x \in (a, b)$ such that $f(x) - P(x) = \frac{f^{(n+1)}(r_x)}{(n+1)!}(x - x_0)(x - x_1)\dots(x - x_n)$

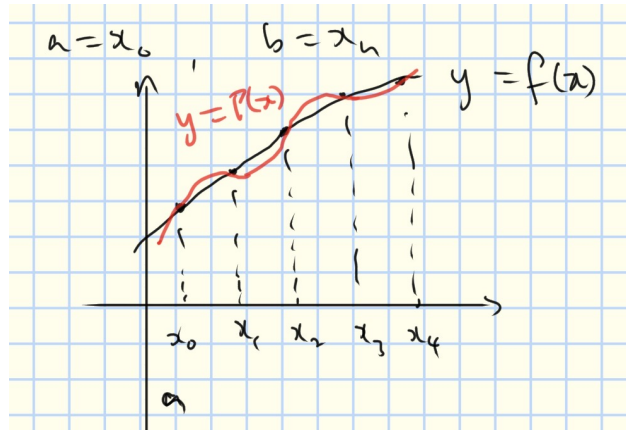


Figure 4: Error Estimate

$$(x_1) \dots (x - x_n)$$