Lecture 11: Error Estimate Continuation (02/04/2026)

So,

$$
\begin{aligned}
f(x) - P(x) &= \frac{f^{n+1}(r_x)}{(n+1)!}(x - x_0)\dots(x - x_n) \\
|f(x) - P(x)| &= \frac{|f^{n+1}(r_x)|}{(n+1)!}|x - x_0||x - x_1|\dots|x - x_n| \\
&\leq \frac{\max\limits_{[x_0,x_n]} |f^{(n+1)}|}{(n+1)!}|x - x_0||x - x_1|\dots|x - x_n|
\end{aligned}
$$

Assume that $x_0, x_1, x_2, \dots, x_n$ are evenly spread,

$$
x_1 = x_0 + h,\ x_2 = x_1 + h = x_0 + 2h, \dots, x_n = x_0 + nh
$$

$$
a = x_0,\ b = x_n
$$

$$
h = \frac{b - a}{h}
$$

$$
|x - x_0| \leq h
$$

$$
|x - x_1| \leq h
$$

$$
|x - x_2| \leq 2h
$$

$$
|x - x_3| \leq 3h
$$

$$
\vdots
$$

$$
|x - x_n| \leq nh
$$

$$|x-x_0||x-x_1|\dots|x-x_n| \le h(h)(2h)\dots(nh) = h^{n+1}n!$$

$$x \in (x_1, x_2)$$

$$|x-x_0| \le 2h$$

$$|x-x_1| \le h$$

$$|x-x_2| \le h$$

$$|x-x_3| \le 2h$$

$$\vdots$$

$$|x-x_n| \le (n-1)h$$

$$|x-x_0|\dots|x-x_n| \le h^{n+1}\ 2 \times (n-1)!$$

$$|f(x)-P(x) \le \frac{\max\limits_{[x_0,x_n]}|f^{(n+1)}|}{(n+1)!} \underbrace{|x-x_0||x-x_1|\dots|x-x_n|}_{\le\ h^{n+1}n!}$$

$$\le \frac{\max\limits_{[a,b]}|f^{(n+1)}|}{n+1} h^{n+1}$$

<u>Conclusion</u>

$$\epsilon = (\max_{[a,b]}|f^{(n+1)}|)\ \frac{h^{n+1}}{n+1}$$