Lecture 22: Gauss-Legendre Continuation (03/09/2026)

$$\int_a^b f(x)\,dx \approx w_1 f(x_1) + \cdots + w_n f(x_n)$$

We want to choose $w_1, \ldots, w_n$ (weights) and $x_1, \ldots, x_n$ (sample points).

We want this to be exact for

$$f(x) = 1, x, x^2, \ldots, x^{2n-1}.$$

Case $[a, b] = [-1, 1]$

1-point rule ($n = 1$)

$$\int_{-1}^1 f(x)\,dx \approx w_1 f(x_1)$$

We want this to be exact for $f(x) = 1$ and $f(x) = x$.

When $f(x) = 1$:

$$\int_{-1}^1 1\,dx = w_1 \quad \Rightarrow \quad w_1 = 2.$$

When $f(x) = x$:

$$\int_{-1}^1 x\,dx = w_1 f(x_1) = w_1 x_1 \quad \Rightarrow \quad 0 = w_1 x_1 \Rightarrow x_1 = 0.$$

Therefore,

$$\int_{-1}^1 f(x)\,dx \approx 2f(0).$$

Gauss-Legendre rule of 1 point.

Change of Variable

If the interval is not $[-1,1]$, change variables.

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(a + \frac{b-a}{2}(t+1)\right)\frac{b-a}{2}\,dt$$

where

$$x = a + \frac{b-a}{2}(t+1).$$

2-point rule

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2)$$

We want this to be exact for $f(x) = 1, x, x^2, x^3$.

When $f(x) = 1$:

$$2 = w_1 + w_2.$$

When $f(x) = x$:

$$0 = w_1 x_1 + w_2 x_2.$$

When $f(x) = x^2$:

$$\frac{2}{3} = w_1 x_1^2 + w_2 x_2^2.$$

When $f(x) = x^3$:

$$0 = w_1 x_1^3 + w_2 x_2^3.$$

Solving,

$$w_1 = w_2 = 1, \quad x_1 = -\frac{1}{\sqrt{3}}, \quad x_2 = \frac{1}{\sqrt{3}}.$$

Therefore,

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$