Lecture 27: Boundary Value Problems and Finite Difference Method (03/30/26)

0.23 Example

$$y'' + y' + y = 0$$

with boundary conditions:

$$y(0) = 1, \quad y(1) = 2$$

Definition: A boundary value problem (BVP) specifies conditions at more than one point (unlike initial value problems).

1 Discretization of Interval

Let:

$$x_0 = 0, x_1, x_2, \ldots, x_n = 1$$

Step size:

$$h = x_{k+1} - x_k$$

Approximate:

$$y_k \approx y(x_k)$$

2 Finite Difference Approximations

2.1 First Derivative

$$y'(x_k) \approx \frac{y_{k+1} - y_k}{h}$$

2.2 Second Derivative

$$y''(x_k) \approx \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2}$$

3 Substitute into Equation

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + \frac{y_{k+1} - y_k}{h} + y_k = 0$$

Multiply by h^2:

$$y_{k+1} - 2y_k + y_{k-1} + h(y_{k+1} - y_k) + h^2 y_k = 0$$

4 Rearranged Equation

$$(1 + h)y_{k+1} + (-2 - h + h^2)y_k + y_{k-1} = 0$$

5 Unknowns

$$y_1, y_2, \ldots, y_{n-1}$$

Note: y_0 and y_n are known from boundary conditions.

6 Linear System Form

We obtain a system:

$$AY = B$$

where:

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Matrix A is tridiagonal:

$$A = \begin{bmatrix} a_1 & c & 0 & \cdots & 0 \\ b & a_2 & c & \cdots & 0 \\ 0 & b & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & c \\ 0 & 0 & 0 & b & a_{n-1} \end{bmatrix}$$

7 Solution

$$Y = A^{-1}B$$

Key Idea: We converted a differential equation into a system of linear equations.

8 Practice Problem

$$y'' - xy' + y = x^2$$

with:

$$y(0) = 2, \quad y(1) = 1$$

Grid:

$$h = 0.25$$

Points:

$$x_0, x_1, x_2, x_3, x_4$$

Unknowns:

$$y_1, y_2, y_3$$

9 Discrete Equation

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} - x_k \frac{y_{k+1} - y_k}{h} + y_k = x_k^2$$

Multiply by h^2:

$$y_{k+1} - 2y_k + y_{k-1} - x_k h(y_{k+1} - y_k) + h^2 y_k = x_k^2 h^2$$

10 Final Form

$$y_{k+1}(1 - x_k h) + y_k(-2 + x_k h + h^2) + y_{k-1} = x_k^2 h^2$$