Lecture 4

Friday, January 16, 2026 4:58 AM

Newton-Ralphson method (also known as Newton's method) is a fast numerical method to solve the equation $f(x) = 0$. Its convergence is, however, not guaranteed. The idea is as follows.

We start with an initial guess x_0 for solution. At $x = x_0$, the function f is approximated by a line, the best of which is the tangent line at x_0:

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

The solution to $L(x) = 0$ is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. Now x_1 becomes the next guess for solution.
Then we compute the next guess, x_2, in a similar way. At $x = x_1$, the function f is approximated by a line, the best of which is the tangent line at x_1:

$$L(x) = f(x_1) + f'(x_1)(x - x_1)$$

The solution to $L(x) = 0$ is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. The next guess x_3 is calculated similarly. You get a recursive equation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Why is Newton's method not always successful?
$f'(x)$ could be zero, which immediately terminates the iteration with an error.
$f'(x)$ could be close to zero, making the fraction $f(x_n)/f'(x_n)$ very large. This drives x_{n+1} away from the desirable solution.

Example: find $\sqrt{2}$ using Newton-Raphson method with 3 iterations.
$\sqrt{2}$ is the positive root of $f(x) = x^2 - 2$.
Initial guess: $x_0 = 1$
Recursion formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$$

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0} = \frac{1}{2} + 1 = \frac{3}{2}$$

$$x_2 = \frac{x_1}{2} + \frac{1}{x_1} = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$$

$$x_3 = \frac{x_2}{2} + \frac{1}{x_2} = \frac{17}{24} + \frac{12}{17} = \frac{577}{408} \approx 1.41421568$$

newton.mlx

We solve the equation $f(x) = 0$ using Newton-Raphson method.

```
format long
x0 = 1;
x = x0;
for k = 1:5
    x = x - f(x)/df(x)
end
```

Define the function f:

```
function y = f(x)
    y = x^2 - 2;
end
```

Compute the derivative f':

```
function y = df(x)
    y = 2*x;
end
```

[Work on the worksheet]