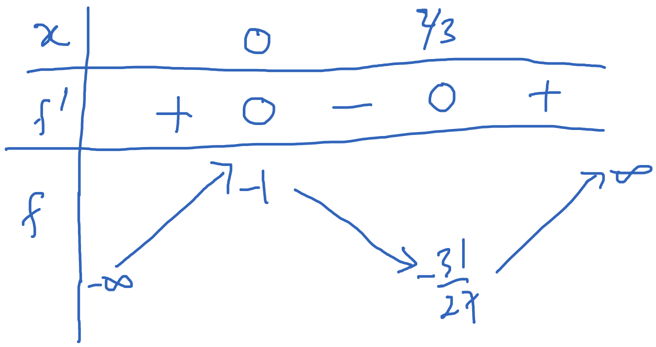
Lecture 5

Wednesday, January 21, 2026 1:57 AM

Fixed point method: $x = \phi(x)$
Iteration: $x_{n+1} = \phi(x_n)$
If the sequence $\{x_n\}$ converges, its limit is a fixed point of ϕ.

Example: Show that $f(x) = x^3 - x^2 - 1$ has only one real root. Then use the fixed point method to find this root correct to 2 decimal places.
Note that $f'(x) = 3x^2 - 2x = x(3x - 2)$. So, f has two critical numbers: $x = 0$ and $x = 3/2$.



The variation table shows that f has only one root, and that root is greater than 2/3. One can better localize this root by observing that $f(1) = -1 < 0$ and $f(2) = 3 > 0$. So, the root must be in between 1 and 2.

There are many ways to turn the root-finding problem $f(x) = 0$ to a fixed point problem $x = \phi(x)$. For example,

$$x = x^3 - x^2 + x - 1$$

$$x^2(x-1) - 1 = 0 \Rightarrow x - 1 = \frac{1}{x^2} \Rightarrow x = 1 + \frac{1}{x^2}$$

If you choose $\phi(x) = x^3 - x^2 + x - 1$, you will see that the recursive sequence $\{x_n\}$ diverges for almost any choice of x_0 between 1 and 2.
If you choose $\phi(x) = 1 + 1/x^2$, you will see that $\{x_n\}$ converges slowly to a limit for almost any choice of x_0 between 1 and 2.

Cobweb diagram is a visual representation of the sequence $\{x_n\}$. Experiment it here:
https://www.geogebra.org/m/QJ79IWCL

In general, there is no guarantee that the recursive sequence $\{x_n\}$ converges, even if fixed points of ϕ exist. However, **if $\phi: I \to I$ and $|\phi'(x)| < 1$ for all $x \in I$ then the sequence $\{x_n\}$ converges.** Here, I is a an interval, such as $[a, b]$, $(a, b]$, $(-\infty, a]$, $(-\infty, \infty)$, ...
Why? Let $\alpha = \max|\phi'(x)| < 1$. Then:

$$|x_{n+1} - x_*| = |\phi(x_n) - \phi(x_*)| = |\phi'(c_n)||x_n - x_*| \le \alpha|x_n - x_*|$$

Thus, $|x_n - x_*| \le \alpha^n |x_0 - x_*|$, which converges to zero as $n \to \infty$.

[Work on the worksheet]