Lecture 6

Friday, January 23, 2026   3:13 AM

We learned some numerical methods to solve the equation $f(x) = 0$:

- Bisection method
- Chord method (also known as *false position* or *linear interpolation* method)
- Newton-Raphson method
- Fixed point method

Which of them is the best? It depends on what you mean by "best". The first two methods guarantee success, but the last two methods don't. Suppose all methods successfully approximate the exact solution. The effectiveness of the method depends on the accuracy, speed, and memory consumption. Newton-Raphson method gives faster convergence, i.e. it takes fewer iterations to achieve a desirable accuracy. However, it can require more calculations for each iteration than the bisection method. A quantity that measures the speed of convergence is the *order (or rate) of convergence*.

Suppose x_* is the exact solution to $f(x) = 0$. Let $\{x_n\}$ be a sequence that approximates x_*. That is, $\lim x_n = x_*$. Let $\epsilon_n = |x_n - x_*|$ be the error at the n'th iteration.
If there exist two constants $C, p > 0$ such that $\epsilon_{n+1} \leq C\epsilon_n^p$ for all n then p is called the *order (or rate) of convergence* of $\{x_n\}$.
If $p = 1$, we say that x_n converges to x_* at a *linear rate*. You can see that the larger p is, the faster ϵ_n goes to 0.

Example: recall that we computed $\sqrt{2}$ numerically using Newton-Raphson method for the equation $x^2 - 2 = 0$. The iteration is

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

What is the order of the convergence $x_n \to \sqrt{2}$? Subtract $\sqrt{2}$ from both sides of the above equation:

$$x_{n+1} - \sqrt{2} = \frac{x_n}{2} + \frac{1}{x_n} - \sqrt{2} = \frac{x_n^2 - 2\sqrt{2}x_n + 2}{2x_n} = \frac{\left(x_n - \sqrt{2}\right)^2}{2x_n}$$

Take the absolute value of both sides:

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2|x_n|} \approx \frac{\epsilon_n^2}{2\sqrt{2}}$$

Thus, $p = 2$ and $C \approx \frac{1}{2\sqrt{2}}$. The convergence rate is $p = 2$.

Example: the order of convergence of the fixed point method $x = \phi(x)$ depends on the function ϕ. The recursive formula is $x_{n+1} = \phi(x_n)$.
Newton-Raphson method is a special case of fixed point method, where $\phi(x) = x - \frac{f(x)}{f'(x)}$. The order of convergence in that case is $p = 2$. However, the order of convergence can be lower or higher if we choose a different ϕ. What is the order of convergence in the worst case (the case that corresponds to the lowest order of convergence)?
Suppose $\max|\phi'(x)| = \alpha < 1$. Let x_* be the exact fixed point of ϕ. By Mean Value Theorem, there exists a number c_n between x_n and x_* such that

$$x_{n+1} - x_* = \phi(x_n) - \phi(x_*) = \phi'(c_n)(x_n - x_*)$$

Take the absolute value of both sides:

$$\epsilon_{n+1} = |\phi'(c_n)|\epsilon_n \leq \alpha\epsilon_n$$

The order of convergence in this case is $p = 1$.

One can find the order of convergence numerically. Note that $\epsilon_{n+1} \approx C\epsilon_n^p$ is equivalent to

$$\ln \epsilon_{n+1} \approx \ln C + p \ln \epsilon_n$$

For sufficiently large n, you can approximate $\epsilon_n = |x_n - x_*|$ by $e_n = |x_n - x_{n-1}|$. Then

$$\ln e_{n+1} \approx \ln C + p \ln e_n$$

The error e_n can be computed. You then create two vectors $Y = [\ln e_{m+1}, \ln e_{m+2}, \dots, \ln e_{m+N}]$ and vector $X = [\ln e_m, \ln e_m, \dots, \ln e_{m+N-1}]$. Then you find the line that best fits N points whose x-coordinates and y-coordinates are given in vectors X and Y. The slope of this line is approximately the value of p and the y-intercept of this line is approximately $\ln C$.

In Matlab, you use the code below to plot the line that best fits and find its slope and y-intersect:

```
f = fit(X,Y,'poly1')
plot(f,X,Y,'*')
```