Lecture 7

Monday, January 26, 2026 5:28 AM

Goal: Lagrange interpolation

Suppose that you have a list of data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$. Can you find a polynomial that passes through all of them?

This question has several variants. For example,
(1) interpolate the given data points by a polynomial
(2) find a polynomial that agrees with a non-polynomial function at a certain point.
(3) approximate a non-polynomial function by a polynomial.

What is the strategy? Suppose $P(x)$ is a polynomial whose graph passes through the points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$. Then we know that $P(x_k) = y_k$ for all $k = 0, 1 \dots, n$. You can see immediately that if $x_k = x_j$ then $y_k = y_j$. Otherwise, there will be no such polynomial. And if it happens that $x_k = x_j$ and $y_k = y_j$ for some $k \neq j$ then the data point (x_k, y_k) is repeated on the list and you can drop the repetition before proceeding.

It would be economical to find a polynomial with *the lowest* degree possible. Higher degrees mean there are more coefficients to find and that is not effective. What can be the lowest degree can P possible have? If $\deg P = k$ then P has $k + 1$ coefficients to be determined. We have $n + 1$ equations to be satisfied. The number of unknowns must be at least the number of equations to be solvable in general. So, we need $k + 1 \geq n + 1$. So, k must be at least n.

Can there be more than one such polynomial of degree n? If $Q(x)$ be such a polynomial then $R = P - Q$ satisfies $R(x_k) = 0$ for all $k = 0, 1, \dots, n$. You see that R has degree $\leq n$ and has $n + 1$ roots, so R must be the constant-zero polynomial. That means P and Q are exactly the same.

In summary, the lowest possible degree of polynomial P is n. And if such a polynomial exists, it is unique. Such a polynomial is called *Lagrange polynomial*. In other words, *Lagrange polynomial is the lowest degree polynomial that interpolates a set of data*. How do we find it?

Let $P_k(x)$ be the product of $x - x_j$ for all $0 \leq j \leq n, j \neq k$. Then $P(x) = y_0 \frac{P_0(x)}{P_0(x_0)} + y_1 \frac{P_1(x)}{P_1(x_1)} + \cdots + y_n \frac{P_n(x)}{P_n(x_n)}$. The polynomials $l_k(x) = \frac{P_k(x)}{P_k(x_k)}$ are called *Lagrange fundamental polynomials*.

Example:
Find the polynomial of degree ≤ 2 passing through $(0, 1), (1, 3), (2, 4)$?

$$l_0(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x-1)(x-2)$$

$$l_1(x) = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -x(x-2)$$

$$l_2(x) = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{1}{2}x(x-1)$$

Therefore, $P(x) = 1 \cdot l_0(x) + 3 \cdot l_1(x) + 4 \cdot l_2(x) = \frac{1}{2}(x-1)(x-2) - 3x(x-2) + 2x(x-1)$.

[work on the worksheet]