Numerical method to find order of convergence

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Let $\{x_n\}$ be a sequence that converges to x_*. You can find the order of convergence numerically. That is to find the number $p > 0$ such that $\epsilon_{n+1} \approx C\epsilon_n^p$, where $\epsilon_n = |x_n - x_*|$ is the error at the n'th step. By taking the logarithm, you can write this equation as

$$\ln \epsilon_{n+1} \approx \ln C + p \ln \epsilon_n \qquad (1)$$

For sufficiently large n (say $n > m$), x_{n-1} is almost equal to x_*. You can approximate

$$\epsilon_n = |x_n - x_*| \approx |x_n - x_{n-1}|$$

This means all error terms $\epsilon_2, \epsilon_3, \ldots$ can be computed approximately from the sequence $x_1, x_2, \ldots$

For some number N, you create two vectors of length N:

$$Y = [Y_1, Y_2, \ldots, Y_N] = [\ln \epsilon_{m+1}, \ln \epsilon_{m+2}, \ldots, \ln \epsilon_{m+N}]$$
$$X = [X_1, X_2, \ldots, X_N] = [\ln \epsilon_m, \ln \epsilon_{m+1}, \ldots, \ln \epsilon_{m+N-1}]$$

Equation (1) implies

$$Y_1 \approx \ln C + pX_1$$
$$Y_2 \approx \ln C + pX_2$$
$$\ldots$$
$$Y_N \approx \ln C + pX_N$$

Thus, you are looking for a line $y = \ln C + px$ that best fits the points (X_1, Y_1), (X_2, Y_2),....(X_N, Y_N). The slope of this line is approximately the value of p (the order of convergence) and the y-intercept of this line is approximately $\ln C$. (You will need to exponentiate the y-intercept to get C.)

In MATLAB 2025, you use the code below to plot the line that best fits and find its slope and y-intersect:

```
f = fit(X,Y,'poly1')
plot(f,X,Y,'*')
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