Worksheet 4/3/2026

$$y'(x_k) = \frac{y_{k+1} - y_k}{h} + O(h) = \frac{y_k - y_{k-1}}{h} + O(h) = \frac{y_{k+1} - y_{k-1}}{2h} + O(h)$$

$$y'(x_k) = \frac{-3y_k + 4y_{k+1} - y_{k+2}}{2h} + O(h^2) = \frac{y_{k-2} - 4y_{k-1} + 3y_k}{2h} + O(h^2) = \frac{y_{k+1} - y_{k-1}}{2h} + O(h^2)$$

$$y''(x_k) = \frac{y_k - 2y_{k+1} + y_{k+2}}{h^2} + O(h) = \frac{y_{k-2} - 2y_{k-1} + y_k}{h^2} + O(h) = \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + O(h)$$

$$y''(x_k) = \frac{2y_k - 5y_{k+1} + 4y_{k+2} - y_{k+3}}{h^2} + O(h^2) = \frac{-y_{k-3} + 4y_{k-2} - 5y_{k-1} + 2y_k}{h^2} + O(h^2) = \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + O(h^2)$$

Consider the boundary value problem $u_{xx} + u_{yy} + u_y + u = x$ on the square $R = [0,1] \times [1,2]$ with boundary condition $u(x, y) = x + y$ for all $(x, y) \in \partial R$.

1) With nodal points $0 = x_0 < x_1 < \cdots < x_n = 1$ and $1 = y_0 < y_1 < \cdots < y_m = 2$, and $u_{jk} = u(x_j, y_k)$, use the boundary conditions to find $u_{jm}, u_{0,m}, u_{n,k}, u_{0,k}$.

2) Derive the difference equations at order $O(h^2)$ for the problem using central difference at all interior grid points.

3) With mesh size $h = 1/3$ for both the x-axis and y-axis, write the system of 4 linear equations satisfied by $u_{j,k}$ where $j, k \in \{1, 2\}$.