Lecture 12: Exact Differential Equations (02/06/2026)

An exact differential equation is one that comes from a single function, often written as $\phi(x, y)$. This means the equation can be viewed as the total derivative of where its partial derivatives with respect to x and y match the given terms.

$$y' = -\frac{\phi x}{\phi y}, \quad \phi = \phi(x, y)$$

Examples

$$(1).\ \phi(x, y) = x + y$$

$$y' = -\frac{1}{1} = -1$$

$$(2).\ \phi(x, y) = e^{xy}$$

$$\phi_x = ye^x$$

$$\phi_y = xe^{xy}$$

$$y' = -\frac{ye^{xy}}{xe^{xy}}$$

$$y' = -\frac{-y}{x}$$

$$(3).\ \phi(x,y) = x^2y + y^3$$

$$\phi_x = 2xy$$

$$\phi_y = x^2 + 3y^2$$

$$y' = -\frac{2xy}{x^2+3y^2}$$

To solve these type of differential equations, we have

$$y' = -\frac{\phi_x}{\phi_y}$$

$$y'\phi_y = -\phi_x$$

$$\phi_x + y'\phi_y = 0$$

$$\frac{d}{dx}[\phi(x,y)] = \phi(x,y)$$

$$x = x(t),\ \ y = y(t)$$

$$\frac{d}{dx}[\phi(x,y)] = \frac{\partial\phi}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial\phi}{\partial y}\cdot\frac{dy}{dx}$$

$$\phi_x + \phi_y y' = 0$$

$$\phi(x,y) = 0$$

Examples

$$(1).\quad y' = -\frac{2y}{2x+3y^2} = -\frac{\phi_x}{\phi_y}$$

$$\phi(x,y) = 2xy + y^3$$

$$2xy + y^3 = C$$

$$\text{If } y(0) = 2, \text{ then}$$

$$2(0)(2) + (2)^3 = C \to C = 8$$

$$2xy + y^3 = 8$$

Solution to the ODE is in implicit form

$$(2).\quad y' = -\frac{x}{y}, \quad y(1) = 2$$

$$\phi(x,y) = x^2 + y^2$$

$$\phi_x = 2x, \quad \phi_y = 2y$$

$$\frac{\phi_x}{\phi_y} = \frac{x}{y}$$

$$y' = -\frac{\phi_x}{\phi_y}$$

$$x^2 + y^2 = C$$

$$(1)^2 + (2)^2 = C \to C = 5$$

$$x^2 + y^2 = 5$$

$$y^2 = 5 - x^2$$

$$y = \pm\sqrt{5 - x^2}$$

$$y = \sqrt{5 - x^2}$$

How do we find ϕ?

$$y' = \frac{f(x,y)}{g(x,y)} = -\frac{\phi_x}{\phi_y}$$

We want :

$\phi_x = -f(x, y) \to \phi_{xy} = -f_y$

$\phi_y = g(x, y) \to \phi_{yx} = g_x$

If $-f_y = g_x$, then $\underline{\phi \text{ exists}}$. Otherwise ϕ doesn't exists.

Example

$$(2xy + 3x^2)dx + (x^2 + 1)dy = 0, \ y(0) = 1$$

$$\underbrace{2xy + 3x^2}_{\phi_x} + \underbrace{(x^2 + 1)y'}_{\phi_y} = 0$$

$$\frac{\partial}{\partial y}(2xy + 3x^2) = \frac{\partial}{\partial x}(x^2 + 1)$$

$$\phi_x = x^2 y + y^3 + C(y)$$

$$\phi_y = x^2 + 0 + C'(y) \to C'(y) = 1$$

$$C(y) = y + C_1$$

$$\phi(x, y) = x^2 y + x^3 + y$$

$$(0)^2(1) + (0)^3 + 1 = 1 \to C = 1$$

$$x^2 y + y + x^3 = 1$$

$$y(x^2 + 1) + x^3 = 1$$

$$y(x^2 + 1) = 1 - x^3$$

$$y = \frac{1 - x^3}{x^2 + 1}$$