Lecture 13: Second Order Linear ODE (02/13/2026)

General form of linear second order ODE is $y'' + p(x)y' + q(x)y = f(x)$; p, q, f are given functions.

Picard Theorem : $y' = f(x, y)$

If f and $\frac{\partial f}{\partial y}$ are continuous then there exists a unique solution.

Apply to $y' + p(x)y = q(x) \rightarrow y' = \underbrace{q(x) - p(x)y}_{f(x,y)}$, $\quad \frac{\partial f}{\partial y} = -p(x)$

If p and q are continuous functions, then f and $\frac{\partial f}{\partial y}$ are also continuous. By Picard's theorem, a solution exists and is unique on a small interval.

Theorem:

If p, q, f are continuous functions, then on an interval I and $a \in I$, then the initial value problem:

$y'' + p(x)y' + q(x)y = f(x)$

$y(a) = \alpha, \quad y'(a) = \beta$

has a unique solution on I.

Theorem:

Let p, q be continuous functions on the interval I. Supposed y_1 and y_2 are two linearly independent solutions to the equation $y'' + p(x)y' + q(x)y = 0$. Then every solution to this ODE is a linear combination of y_1 and y_2.

* Linear combination of two functions y_1 and y_2 is a function $c_1 y_1 + c_2 y_2$.

Ex: $2\sin x + 3\cos x$ is a linear combination of $\sin x$ and $\cos x$.

* Suppose y_1 and y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$, why is

$y_1 + 2y_2$ also a solution?

$$y_1'' + p(x)y_1' + p(x)y_1 = 0$$

$$2(y_2'' + p(x)y_2 + q(x)y_2 = 0) + (2y_2)'' + 2p(x)y_2 + 2p(x)y_2 = 0$$

$$(y_1 + 2y_2)'' + (p(x)(y_1 + 2y_2)'' + (q(x)(y_1 + 2y_2)) = 0$$

$$\text{(true for homogeneous equations } b(x) = 0\text{)}$$

Two functions $y_1, 2y_2$ are linearly independent if and only if $c_1y_1 + c_2y_2 = 0 \quad \forall x$. Then, $c_1 = c_2 = 0$.