Lecture 14: Constant-coefficient second order linear ODE (02/18/2026)

The functions $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independentof the only constants $c_1, c_2, \ldots, c_n$ that satisfy $c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0 \quad \forall x$ are $c_1 = c_2 = \ldots c_n = 0$

Ex 1:

$$y_1 = \cos x, \quad y_2 = \sin x$$

Are they linearly independent?

Let c_1, c_2 be constants such that $c_1 \cos x + c_2 \sin x = 0 \quad \forall x$

Plug in $x = 0$:

$$c_1(1) + c_2(0) = 0$$

$$c_1 = 0$$

Plug in $x = \frac{\pi}{2}$:

$$c_1\left(\frac{\pi}{2}\right) + c_2\left(\frac{\pi}{2}\right) = 0$$

$$c_2(1) = 0$$

$$c_2 = 0$$

Therefore y_1 and y_2 are linearly independent.

Ex 2:

$$y_1 = \cos x,\ y_2 = \sin x,\ y_3 = \cos(x+1)$$

Are y_1, y_2, y_3 linearly independent?

$$y_3 = \cos(x+1) = \cos x \cos 1 - \sin x \sin 1$$

$$= (\cos 1) y_1 + (-\sin 1) y_2$$

$$\text{So, } (\cos 1) y_1 + (-\sin 1) y_2 + (-1) y_3 = 0$$

Ex 3:

$$c_1 \sin x + c_2 e^x = 0 \quad \forall x$$
$$\frac{c_1 \sin x}{e^x} + \frac{c_2 e^x}{e^x} = \frac{0}{e^x}$$
$$\frac{c_1 \sin x}{e^x} + c_2 = 0$$
$$\lim_{x \to \infty} \left(\frac{c_1 \sin x}{e^x} + c_2 \right) = 0$$
$$\frac{\sin x}{e^x} \to 0$$

$$0 + c_2 = 0$$

$$c_2 = 0$$

$$c_1 \sin x = 0$$

$$c_1 = 0$$

Therefore, the functions are linearly independent.

Ex 4:

$$c_1 \sin(x+4) + c_2 \cos(x-2) = 0 \quad \forall x$$

$$\text{Choose } x = \frac{\pi}{2} + 2$$

$$c_1 \sin\left(\frac{\pi}{2} + 6\right) + c_2 \cos\left(\frac{\pi}{2}\right) = 0$$

$$c_1 \sin\left(\frac{\pi}{2} + 6\right) + 0 = 0$$

$$c_1 = 0$$

$$c_2 \cos(x-2) = 0$$

$$\text{Choose } x = 2$$

$$c_2 \cos(0) = 0$$

$$c_2 = 0$$

Therefore, the functions are linearly independent.

If y_1 and y_2 are solutions and y_1 and y_2 are linearly independent, the <u>all</u> solutions to the ODE are the linear combinations of y_1 and y_2.

$$c_1 y_1 + c_2 y_2 = 0 \quad \forall x$$

$$\frac{y_1}{y_2} = -\frac{c_2}{c_1} = \text{constant}$$

If $\frac{y_1}{y_2}$ is not a constant function then y_1 and y_2 are linearly independent.

0.4.1 Homogeneous Second Order Linear ODE with constant coefficients

$$ay'' + by' + cy = 0, \text{ where a,b,c are constants}$$

Seek special solutions of the form:

$$y = e^{rx}$$

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

$$ar^2 e^{rx} + bre^{rx} + ce^{rx}$$

$$(ar^2 + br + c)e^{rx}$$

To make this equal to zero, we choose r such that $ar^2 + br + c = 0$