Lecture 18: Second Order Linear Homogeneous Differential Equation (02/27/2026)

A second order linear homogeneous differential equation with constant coefficients is

$$my'' + cy' + ky = 0$$

where

- m = mass
- c = damping coefficient
- k = spring constant

This equation models a damped mass–spring system.

0.7.3 Characteristic Equation

Assume a solution of the form $y = e^{rx}$

Substituting into the differential equation gives the characteristic equation

$$mr^2 + cr + k = 0$$

Solving this quadratic determines the behavior of the system.

0.7.4 Discriminant

Define

$\Delta = c^2 - 4mk$

The sign of the discriminant determines the type of motion.

0.7.5 Case 1: Overdamped Motion

Condition: $c^2 > 4mk$.

The two distinct real roots are r_1, r_2

The solution is: $y(t) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Behavior: fast decay without oscillation.

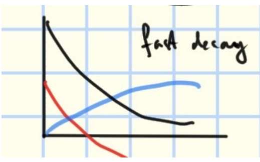

Figure 9: Overdamped Motion

0.7.6 Case 2: Critically Damped Motion

Condition: $c^2 = 4mk$

There is one repeated root: r

The solution is: $y(t) = c_1 e^{rt} + c_2 t e^{rt}$

Behavior: fastest return to equilibrium without oscillation.

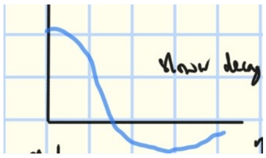

Figure 10: Critically Damped Motion

0.7.7 Case 3: Underdamped Motion

Condition: $c^2 < 4mk$

Complex roots $r = \alpha \pm i\beta$

The solution is: $y(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$

Behavior: oscillations with decreasing amplitude.

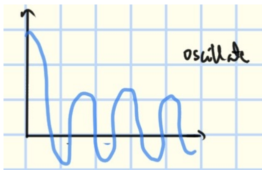

Figure 11: Underdamped Motion

0.7.8 Undamped Motion

If $c = 0$, then the motion is undamped and oscillations continue indefinitely.

Summary of Motion Types

$\Delta = c^2 - 4mk$

- $c^2 > 4mk$ Overdamped (no oscillation)
- $c^2 = 4mk$ Critically damped
- $c^2 < 4mk$ Underdamped (oscillatory decay)

- $c = 0$ Undamped oscillation

0.8 Nonhomogeneous Linear Differential Equation

Consider

$$y'' + p(x)y' + q(x)y = f(x)$$

This is called a nonhomogeneous equation.

Suppose that

- y_p is a particular solution
- y_h is the general solution of the homogeneous equation

Then the general solution is

$$y = y_c + y_p$$

If y_1 and y_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0$$

then

$$y_h = c_1 y_1 + c_2 y_2$$

0.8.1 Method for Solving Nonhomogeneous ODE

Step 1

Solve the homogeneous equation $y'' + p(x)y' + q(x)y = 0$ to obtain $y_c = c_1y_1 + c_2y_2$

Step 2

Find a particular solution y_p.

Step 3

The general solution is $y = y_c + y_p$