Lecture 19: Method of Undetermined Coefficients (03/02/2026)

0.9 Introduction

Consider the second order linear nonhomogeneous differential equation

$$y'' + ay' + by = f(x)$$

To solve this equation we must find:

- the complementary (homogeneous) solution y_c
- a particular solution y_p

The complete solution is

$$y = y_c + y_p$$

General structure of the solution:

$$y = y_c + y_p$$

where

- y_c solves the homogeneous equation
- y_p solves the nonhomogeneous equation

0.10 Method of Undetermined Coefficients

The method of undetermined coefficients is used when the forcing function $f(x)$ has a simple form such as:

- polynomials
- exponentials
- sine or cosine functions
- combinations of the above

The idea is to guess the form of y_p and then determine the unknown constants. Instead of solving directly, we make an intelligent guess for the particular solution and determine the unknown constants by substitution.

Example

Consider

$$y'' + y' + y = x - 3$$

Step 1: Guess the form of the particular solution

Since the forcing function is a polynomial of degree 1, $f(x) = x - 3$

we guess $y_p = Ax + B$

Step 2: Compute derivatives

$y_p' = A$

$y_p'' = 0$

Step 3: Substitute into the differential equation

Substitute into

$$y'' + y' + y$$

$$0 + A + (Ax + B)$$

$$= Ax + (A + B)$$

Step 4: Match coefficients

Set equal to the right-hand side, $x - 3$

Matching coefficients gives

$A = 1$

$A + B = -3 \rightarrow B = -4$

$y_p = x - 4$

When using the method of undetermined coefficients, the guess for y_p depends on the forcing function $f(x)$.

$f(x)$	Guess for y_p
Polynomial of degree n	$A_n x^n + \cdots + A_0$
e^{ax}	Ae^{ax}
$\cos(ax), \sin(ax)$	$A\cos(ax) + B\sin(ax)$
Polynomial $\times e^{ax}$	Polynomial $\times e^{ax}$

Sometimes, the guessed function already appears in the homogeneous solution. In that case, we must multiply the guess by x until it becomes linearly independent. If the guess for y_p duplicates part of y_c, multiply the guess by x.

0.11 Summary

To solve

$$y'' + ay' + by = f(x)$$

follow these steps:

1. Solve the homogeneous equation $y'' + ay' + by = 0$ to obtain the complementary solution y_c.
2. Guess the form of the particular solution y_p using the structure of $f(x)$.
3. Substitute the guess into the differential equation.
4. Solve for the unknown constants.
5. Write the full solution $y = y_c + y_p$