Lecture 20: Advanced Undetermined Coefficients (03/04/2026)

0.12 Introduction

Consider the nonhomogeneous differential equation $y'' - 2y' + y = e^x$ We will use the method of undetermined coefficients to find a particular solution. The full solution will have the form $y = y_c + y_p$ where

- y_c is the complementary (homogeneous) solution
- y_p is a particular solution

0.13 Step 1: Solve the Homogeneous Equation

Solve

$$y'' - 2y' + y = 0$$

$$r^2 - 2r + 1 = 0$$

$$(r - 1)^2 = 0$$

Thus

$$r = 1$$

Since the root is repeated, the complementary solution is

$$y_c = c_1 e^x + c_2 x e^x$$

0.14 Step 2: First Guess for the Particular Solution

The forcing function is $f(x) = e^x$. A natural guess would be $y_p = Ae^x$

However, this function already appears in the complementary solution. If the guessed function already appears in the homogeneous solution, it cannot produce a new independent solution. Therefore, we must modify the guess.

0.15 Adjustment Rule

When the guess duplicates part of the homogeneous solution, multiply the guess by x.

Since e^x and xe^x already appear in y_c, we multiply again by x.

Thus the correct guess is: $y_p = Ax^2 e^x$

If the guess duplicates the complementary solution, multiply by x until the guess becomes linearly independent.

0.16 Step 3: Compute Derivatives

$$y_p = Ax^2e^x$$

$$y_p' = A(2xe^x + x^2e^x)$$

$$y_p'' = A(2e^x + 4xe^x + x^2e^x)$$

0.17 Step 4: Substitute Into the Differential Equation

Substitute into

$$y'' - 2y' + y$$

and solve for the constant A. After simplifying, we obtain the value of A and therefore determine y_p.

Once y_p is found, the complete solution is as follows.

$y = y_c + y_p$ where $y_c = c_1e^x + c_2xe^x$

0.18 Common Guess Table

Forcing Function $f(x)$	Guess for y_p
Polynomial	Polynomial of same degree
e^{ax}	Ae^{ax}
$\sin(ax), \cos(ax)$	$A\cos(ax) + B\sin(ax)$
Polynomial $\times$ e^{ax}	Polynomial $\times$ e^{ax}

0.19 Summary of the Method

To solve

$$y'' + ay' + by = f(x)$$

follow these steps:

1. Solve the homogeneous equation to obtain y_c.
2. Guess the form of y_p based on $f(x)$.
3. If the guess duplicates y_c, multiply by x.
4. Substitute the guess into the differential equation.
5. Solve for the unknown constants.
6. Write the complete solution $y = y_c + y_p$