Lecture 21: Undetermined Coefficients Continuation (03/06/2026)

0.20 Superposition Principle

Let L be a linear differential operator. If y_1 and y_2 satisfy

$$L(y_1) = f_1, \quad L(y_2) = f_2,$$

then

$$L(y_1 + y_2) = f_1 + f_2.$$

Example:

$$L = \frac{d^2}{dx^2} + 2\frac{d}{dx} + \frac{d^0}{dx^0}$$

$$L(y) = y'' + 2y' + y$$

$$L(z) = z'' + 2z' + z$$

$$L(y) + L(z) = (y+z)'' + 2(y+z)' + (y+z) = L(y+z)$$

So L is a linear differential operator.

0.21 Application

$$y'' + y' - 2y = e^x + x^2$$

1. Solve the homogeneous equation:

$$y'' + y' - 2y = 0$$

2. Find a particular solution y_p:

$$y'' + y' - 2y = e^x + x^2$$

   Find particular solutions:

$$y_p^{(1)} \text{ for } e^x, \quad y_p^{(2)} \text{ for } x^2$$

   Then

$$y_p = y_p^{(1)} + y_p^{(2)}$$

Solving

For $y'' + y' - 2y = e^x$:

$$y_p = Ae^x, \quad y_p' = Ae^x, \quad y_p'' = Ae^x$$

$$Ae^x + Ae^x - 2Ae^x = e^x$$

$$-2A = 1 \Rightarrow A = -\frac{1}{2}$$

$$y_p^{(1)} = -\frac{1}{2}e^x$$

For $y'' + y' - 2y = x^2$:

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B, \quad y_p'' = 2A$$

Substitute:

$$2A + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

Solve coefficients:

$$A = -\frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = \frac{3}{2}$$

$$y_p^{(2)} = -\frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{2}$$

Homogeneous Solution

$$r^2 + r - 2 = 0 \Rightarrow (r + 2)(r - 1) = 0$$

$$r = -2, 1$$

$$y_c = c_1 e^{-2x} + c_2 e^x$$

General Solution

$$y(x) = y_c + y_p$$

$$y(x) = c_1 e^{-2x} + c_2 e^x - \frac{1}{2}e^x - \frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{2}$$

Variation of Parameters Method

Solve:

$$y'' + p(x)y' + q(x)y = f(x)$$

Assume two independent solutions y_1, y_2. Let:

$$y = u_1 y_1 + u_2 y_2$$

Conditions:

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(x) \end{cases}$$

Example

$$y'' - 2y' + y = e^x$$

Homogeneous solutions:

$$y_1 = e^x, \quad y_2 = xe^x$$

$$y = u_1 e^x + u_2 xe^x$$

$$\begin{cases} u_1' e^x + u_2' xe^x = 0 \\ u_1' e^x + u_2'(e^x + xe^x) = e^x \end{cases}$$

Solve:

$$u_2' = 1, \quad u_1' = -x$$

$$u_2 = x + c_2, \quad u_1 = -\frac{x^2}{2} + c_1$$

$$y = \left(-\frac{x^2}{2} + c_1\right) e^x + (x + c_2)xe^x$$