# Lecture 22: Power Series (03/09/2026)

A power series is an infinite sum

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

This is called a power series centered at $x_0$.

A power series is a function. What is the domain of this function?

**Interval of convergence**

- $R$: radius of convergence.
- The set of all $x$ such that the series converges is called the interval of convergence.
- It can be $(x_0 - R, x_0 + R)$, $[x_0 - R, x_0 + R)$, $(x_0 - R, x_0 + R]$, or $[x_0 - R, x_0 + R]$.
- We find the radius of convergence; the endpoints may or may not converge.

## 0.22 Ratio Test

Consider the sequence $\{b_k\} = b_0, b_1, b_2, \ldots$.

Suppose that

$$L = \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right|$$

Then the series $\sum b_n$ converges if $L < 1$ and diverges if $L > 1$.

## 0.23 Root Test

Everything is the same as the Ratio Test except that

$$L = \lim_{n\to\infty} \sqrt[n]{|b_n|}$$

## Example

Find the interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{k^2}(x-1)^k$$

For $x = 4$:

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{k^2}(3)^k = \sum_{k=1}^{\infty} \frac{(-6)^k}{k^2}$$

$$= -\frac{6}{1^2} + \frac{(-6)^2}{2^2} + \frac{(-6)^3}{3^2} + \cdots$$

diverges.

Let

$$b_k = \frac{(-2)^k}{k^2}(x-1)^k.$$

We compute:

$$\frac{b_{k+1}}{b_k} = \frac{(-2)^{k+1}}{(k+1)^2}(x-1)^{k+1} \cdot \frac{k^2}{(-2)^k(x-1)^k}$$

$$= -2 \cdot \frac{k^2}{(k+1)^2}(x-1)$$

So,

$$\left|\frac{b_{k+1}}{b_k}\right| = 2 \cdot \frac{k^2}{(k+1)^2}|x-1|$$

$$\lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right| = 2|x-1| \lim_{k\to\infty} \frac{k^2}{(k+1)^2}$$

$$= 2|x-1| \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^2$$

$$= 2|x-1|(1)^2 = 2|x-1|$$

Let $L = 2|x-1|$.

## Solve $L < 1$

$$2|x-1| < 1$$

$$|x-1| < \frac{1}{2}$$

$$-\frac{1}{2} < x - 1 < \frac{1}{2}$$

$$\frac{1}{2} < x < \frac{3}{2}$$

## Solve $L > 1$

$$x < \frac{1}{2} \quad \text{or} \quad x > \frac{3}{2}$$

Thus, radius of convergence:

$$R = \frac{1}{2}$$

## Check Endpoints

At $x = \frac{1}{2}$:

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{k^2} \left(-\frac{1}{2}\right)^k = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges (p-series with $p = 2$).

At $x = \frac{3}{2}$:

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{k^2} \left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

converges by the alternating series test.

## Final Answer

Interval of convergence is

$$\left[\frac{1}{2}, \frac{3}{2}\right].$$