Lecture 25: Power Series Method for Differential Equations (03/16/26)

Example

$$y' = y, \quad y(0) = 1$$

We now use the method of power series.

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

We need to find $a_0, a_1, a_2, \ldots$

$$y(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + \cdots$$

$$1 = a_0$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Equate the power of x^n:

$$a_{n+1}(n+1) = a_n$$

$$a_{n+1} = \frac{a_n}{n+1}$$

(recursive formula)

$$a_0 = 1, \quad a_1 = \frac{a_0}{1} = 1, \quad a_2 = \frac{a_1}{2} = \frac{1}{2}, \quad a_3 = \frac{a_2}{3} = \frac{1}{2 \cdot 3}, \ldots$$

$$a_n = \frac{1}{n!}$$

Therefore,

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This is the power series for e^x.

Example

$$y'' + y' + xy = e^x, \quad y(0) = 1, \quad y'(0) = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n$$

$$y'' = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$\text{LHS} = \sum_{n=0}^{\infty} b_n x^n + \sum_{n=0}^{\infty} c_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$$

Break off the first terms so the starting indices match:

$$= b_0 + \sum_{n=1}^{\infty} b_n x^n + c_0 + \sum_{n=1}^{\infty} c_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$= b_0 + c_0 + \sum_{n=1}^{\infty}(b_n + c_n + a_{n-1})x^n$$

Substitute expressions:

$$= a_1 + 2a_2 + \sum_{n=1}^{\infty}\left[a_{n+1}(n+1) + a_{n+2}(n+2)(n+1) + a_{n-1}\right]x^n$$

Right-hand side:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}x^n$$

Equate coefficients

$$1 = a_1 + 2a_2$$

$$\frac{1}{n!} = a_{n+1}(n+1) + a_{n+2}(n+2)(n+1) + a_{n-1}$$

(recursive formula)

Find $a_0, a_1, a_2, a_3, \ldots$