Lecture 27: Systems of ODEs and Matrix Form (03/30/2026)

We can transform a system of ordinary differential equations (ODEs) into matrix form.

0.25 Example

$$\begin{cases} x_1' = 2x_1 + 3x_2 + t \\ x_2' = x_1 + 2x_2 \end{cases}$$

0.26 Matrix Representation

Define the vector:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then the system becomes:

$$X' = AX + b$$

where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

0.27 Initial Condition

$$X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

0.28 Definition

A system of first-order ODEs can be written compactly as:

$$X'(t) = AX(t) + b(t)$$

Explanation: This form allows us to use linear algebra tools such as eigenvalues and eigenvectors to analyze solutions.

1 Higher Order ODE as System

We convert a higher-order ODE into a system of first-order equations.

1.1 Example

$$y^{(4)} + 3y'' + y' + 5y = x^2$$

with initial conditions:

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 2, \quad y'''(0) = 3$$

1.2 Step 1: Define Variables

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \quad y_4 = y'''$$

1.3 Step 2: Rewrite System

$$y_1' = y_2$$

$$y_2' = y_3$$

$$y_3' = y_4$$

$$y_4' = -5y_1 - 3y_3 - y_2 + x^2$$

1.4 Matrix Form

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x^2 \end{bmatrix}$$

1.5 Initial Vector

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

1.6 Key Idea

Any nth-order ODE can be rewritten as a system of n first-order equations.