Lecture 4

Friday, January 16, 2026 4:19 AM

First ODE of the form $y' = f(x, y)$ can be solved "visually". The idea is as follows.

The solution is a curve $y = y(x)$. As this curve passes through the point (a, b), its slope at this point is $y'(a) = f(a, b)$. At any point (a, b) on the plane, you draw a little line segment with slope $f(a, b)$. What you obtain is a *slope field* (also called *direction field*). A solution curve $y = y(x)$ is a curve that is tangent to every little line segment that it touches.

Examples:
$y' = y$
$y' = x$
$y' = x + y$

To sketch a slope field $y' = y$ on Matlab, run the following code:

```
[X, Y] = meshgrid(-2:0.1:2, -1:0.2:4);
F = Y;
quiver(X, Y, ones(size(F)), F)
axis tight, xlabel('x'), ylabel('y')
title('Slope field for y'' = y')
```

To solve the initial value problem $y' = y, y(0) = 2$ using Chebfun, run the following code:

```
L = chebop(0,3);
L.op = @(x,y) diff(y) - y;
L.lbc = 2;
y = L\0;
plot(y, [-2,3])
```

Picard's theorem of existence and uniqueness:
Suppose that f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R = (a, b) \times (c, d)$. Then for any point $(x_0, y_0) \in R$, there exists uniquely a function $y = y(x)$ satisfying $y' = f(x, y)$ and $y(x_0) = y_0$ on a small interval $(x_0 - \delta, x_0 + \delta)$.

Examples:
1) $y' = y^2, y(0) = 1$
In this case, the function $f(x, y) = y^2$ is continuous everywhere and its partial derivatives are also defined everywhere. However, the solution $y(x) = \frac{1}{1-x}$ only exists on $(-\infty, 1)$.

2) $y' = 2\sqrt{|y|}, y(0) = 0.$
In this case, the function $f(x, y) = 2\sqrt{|y|}$ is continuous everywhere but the partial derivative $\frac{\partial f}{\partial y}$ doesn't exist at (0,0). Picard's theorem is not applicable. You can check that the initial value problem has infinitely many solutions: the constant $y \equiv 0$ is a solution, and any function of the form

$$y(x) = \begin{cases} (x-c)^2 & \text{if} \quad x \geq c \\ 0 & \text{if} \quad x < c \end{cases}$$

where $c \geq 0$, is also a solution. You can see this phenomenon on the slope field.