Lecture 5

Wednesday, January 21, 2026 1:57 AM

Picard's theorem of existence and uniqueness:

Suppose that f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R = (a, b) \times (c, d)$. Then for any point $(x_0, y_0) \in R$, there exists uniquely a function $y = y(x)$ satisfying $y' = f(x, y)$ and $y(x_0) = y_0$ on a small interval $(x_0 - \delta, x_0 + \delta)$.

Examples:

1) $y' = y^2, y(0) = 1$

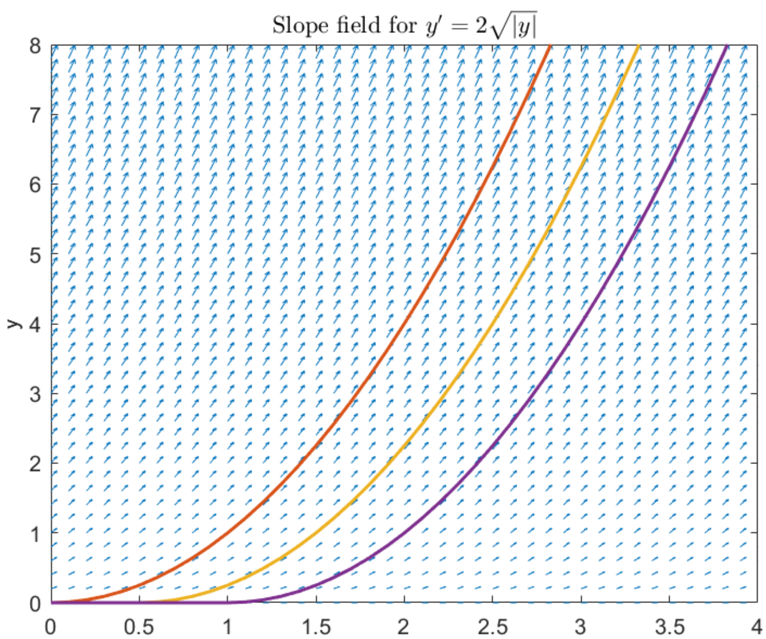
In this case, the function $f(x, y) = y^2$ is continuous everywhere and its partial derivatives are also defined everywhere. However, the solution $y(x) = \frac{1}{1-x}$ only exists on $(-\infty, 1)$.

2) $y' = 2\sqrt{|y|}, y(0) = 0$.

In this case, the function $f(x, y) = 2\sqrt{|y|}$ is continuous everywhere but the partial derivative $\frac{\partial f}{\partial y}$ doesn't exist at (0,0). Picard's theorem is not applicable. You can check that the initial value problem has infinitely many solutions: the constant $y \equiv 0$ is a solution, and any function of the form

$$y(x) = \begin{cases} (x-c)^2 & \text{if} \quad x \geq c \\ 0 & \text{if} \quad x < c \end{cases}$$

where $c \geq 0$, is also a solution. You can see this phenomenon on the slope field.



Matlab Live script (.mlx file):

Plot the slope field (direction field) of the differential equation $y' = 2\sqrt{|y|}$:

```
[X, Y] = meshgrid(0:0.1:4, -1:0.2:8);
F = 2*sqrt(abs(Y));
quiver(X, Y, ones(size(F)), F)
axis tight, xlabel('x'), ylabel('y')
title('Slope field for $y'' = 2\sqrt{|y|}$','Interpreter','latex')
```

The initial value problem $y' = 2\sqrt{|y|}$, $y(0) = 0$ has infinitely many solutions, including $y(x) = \max(x - c, 0)^2$ where $c \geq 0$ is any constant. Now show several solutions on the slope field:

```
hold on
clist = [0 0.5 1];
for k = 1:3
    xx = 0:0.1:4;
    yy = max(0,xx - clist(k)).^2;
    plot(xx,yy,LineWidth=1.5), axis([0 4 0 8])
end
```

Now let's try to solve the initial value problem $y' = 2\sqrt{|y|}$, $y(0) = 0$ using Chebfun toolbox.

```
L = chebop(0,3);
L.op = @(x,y) diff(y) - 2*sqrt(abs(y));
L.lbc = 0;
y = L\0;
```

You will see Chebfun only gives you the simplest solution, which is the constant 0.

```
hold off
plot(y,[-2,3])
```