Lecture 7

Monday, January 26, 2026 12:23 PM

Integrating factor method for linear first order ODE $y' + p(x)y = f(x)$

Analysis: multiply both sides by a function $r(x)$:

$$r(x)y' + r(x)p(x)y = r(x)f(x)$$

Choose function $r(x)$ such that the left hand side is an exact derivative (using product rule).
It turns out that $r'(x) = r(x)p(x)$, which imples $r(x) = e^{\int p(x)dx}$.
The differential equation becomes $(r(x)y)' = r(x)f(x)$.
Integrate both sides:

$$r(x)y = \int r(x)f(x)dx$$

Then divide both sides by $r(x)$ to get the solution:

$$y = \frac{1}{r(x)}\int r(x)f(x)dx$$

Summary of method:

- Compute the integrating factor $r(x) = e^{\int p(x)dx}$. Choose any constant of integration.
- Use the formula of solution $y = \frac{1}{r(x)}\int r(x)f(x)dx$. A constant of integration will appear in the integral.

Examples:

$x^2y' + xy = 1, y(1) = 2$
$x^2y' + xy = 1, y(-1) = 2$

[work on the worksheet]

Suitable substitution $u = u(x, y)$ can help solve certain types of differential equations:

Bernoulli equation

$$y' + p(x)y = q(x)y^n.$$

Here, $n \neq 1$ but doesn't have to be an integer. The strategy is to divide both sides by y^n and then use the substitution $u = y^{1-n}$.

$$y'y^{-n} + p(x)y^{1-n} = q(x)$$

which is equivalent to $\frac{1}{n-1}u' + p(x)u = q(x)$, which is equivalent to

$$u' + (n-1)p(x)u = (n-1)q(x)$$

This is a first-order linear ODE, which can be solve by the integrating factor method. Once you get u, you can find y using the relation $u = y^{1-n}$.

Homogeneous equation

$$y' = f\left(\frac{y}{x}\right)$$

The strategy is to use the substitution $u = \frac{y}{x}$. We have $y = ux$, so $y' = u'x + u$. The differential equation becomes

$$u'x + u = f(u)$$

which is equivalent to $u' = \frac{f(u)-u}{x}$.
This is a separable equation and you can proceed to solve for u. Once you have u, you get $y = ux$.