

## Lecture 8: Homogeneous Equations (01/28/2026)

### *Examples*

$$(1). y' = \frac{xy}{x^2 + y^2} = \frac{\frac{y}{x}}{1 + \frac{y^2}{x^2}} = f\left(\frac{y}{x}\right)$$

$$\text{where } f(t) = \frac{t}{1 + t^2}$$

$$(2). y' = \ln(x) - \ln(y)$$

$$y' = \ln \frac{x}{y} = -\ln\left(\frac{y}{x}\right) = f\left(\frac{y}{x}\right)$$

$$\text{where } f(t) = -\ln(t)$$

### Strategy

$$u = \frac{y}{x}$$

$$y = ux$$

$$y' = u'x + ux \text{ (product rule)}$$

$$= u'x + u$$

### Differentiate Equation

$$u'x + u = f(u)$$

$$u'x = f(u) - u$$

$$u' = \frac{f(u) - u}{x} \text{ (separable equation)}$$

**Example**

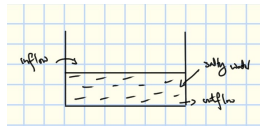


Figure 4: Mixture

$$V_o = 500\text{gal}$$

$$y_o = 0.1\text{lbs/gal} \text{ (initial salt content)}$$

$$\text{pump in fresh water} = 3\text{gal/min}$$

$$\text{pump out} = 2\text{gal/min}$$

Find the salt content at time  $t$ .

The unknown is  $y(t)$  : salt amount at time  $t$  and  $y(t)$  has unit lbs. We need to find a differential equation to solve for  $y$ .

$$\begin{aligned} \frac{dy}{dt} &= \text{instantaneous rate of } y \text{ at time } t. \\ &= (\text{salt coming in}) - (\text{salt coming out}) \\ &= 0 - \underbrace{2}_{\text{gal/min}} \times \underbrace{\frac{y}{V(t)}}_{\text{lbs/gal}} \end{aligned}$$

$\frac{y(t)}{V(t)}$  : amount of salt in a gallon in the tank at time  $t$ .

$$y' = -2 \times \frac{y}{V(t)}$$

$V(t)$  = Volume of the liquid at time  $t$

$$V'(t) = 3 - 2 = 1 \text{ gal/min}$$

$$V(t) = t + C$$

$$V(t) = 500, \text{ so } C = 500$$

$$V(t) = t + 500$$

$$y' = \frac{2y}{t + 500}$$

$$y(0) = 500 \times 0.1 = 50 \text{ lbs}$$

Solve  $y = y(t)$

$$c(t) = \frac{y(t)}{V(t)} = \frac{y(t)}{t + 500}$$