NS 10000001 001

Math 213 Practice Midterm 2

November, 2019

Name: Key

Section:

Instructor:

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Instructions

A) Do not write on the barcode area at the top of each page, or near the four circles on each page.

B) Write your name, section number, and instructor in the space provided, and **COMPLETELY FILL IN** the correct boxes for your BYU ID and for the correct answers in the multiple choice section.

C) Multiple choice questions that are marked with a ♣ may have more than one correct answer. You should mark all correct answers. All other questions have only one correct answer.

D) Multiple choice questions are worth 3 points each.

E) For questions which require a written answer, show all your work in the space provided and justify your answer. Simplify your answers where possible.

F) No books, notes, or calculators are allowed.

G) Do not talk about the test with other students until after the exam period is over.

Part I: Multiple Choice Questions: *(3 points each) Questions marked with a ♣ may have more than one correct answer. Mark **all** correct answers. The other questions have one correct answer. Choose the best answer for each multiple choice question. Fill in the box completely for each correct answer, but DO NOT leave any marks in the other boxes.*

1 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects $\mathbf{x}$ onto the y-axis. What are the eigenvalues of the standard matrix $[T]$?

- □ 0,2
- □ 0,1,−1
- □ 1,2
- □ 1,−1
- □ 0,−1
- ■ 0,1
- □ 0,1,2

2 ♣ Suppose A is an 5×5 diagonalizable matrix, with eigenvalues 0, 0, 1, 1, and 2 (counted with multiplicity). Which of the following statements must be true? Mark all that apply.

- □ The vector $\mathbf{v} = \mathbf{0}$ is an eigenvector corresponding to eigenvalue $\lambda = 0$.
- □ A is equal to a matrix with $0, 0, 1, 1$, and 2 on the diagonal and 0 everywhere else.
- ■ rank $A = 3$
- ■ nullity $A = 2$
- □ A is invertible.
- □ The dimension of the $\lambda = 1$ eigenspace is equal to 1.
- ■ $\det A = 0$

3 ♣ Let

$$A = \begin{bmatrix} -4 & 7 & -7 \\ -5 & 7 & -6 \\ -5 & 4 & -3 \end{bmatrix}.$$

Which of the following vectors are eigenvectors of A? Mark all that apply.

- ☐ $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$
- ■ $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$
- ■ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- ■ $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

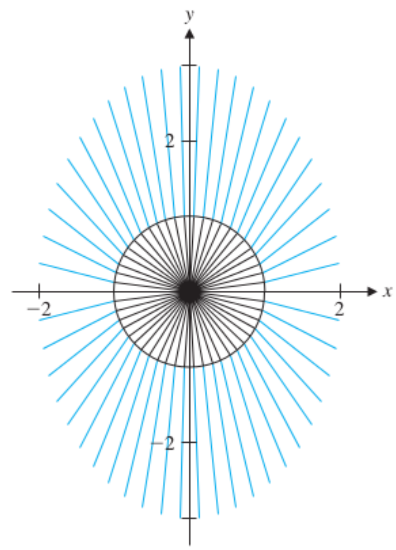
4 Compute $\det \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$.

- ☐ -11
- ☐ 13
- ☐ 1
- ☐ 0
- ☐ -1
- ■ 5
- ☐ -5

5 ♣ Let A and B be $n \times n$ matrices and k a constant. Which of the following is always correct? Mark all that apply.

- [x] $\det(A^T) = \det(A)$
- [x] $\det(AB) = \det(A)\det(B)$
- [ ] $\det(kA) = k\det(A)$
- [ ] $\det(A^{-1}) = \det(A)$
- [ ] $\det(I_n) = n$
- [ ] If A is row equivalent to B, then $\det(A) = \det(B)$
- [x] If $A^2 = I_n$, $\det(A) = \pm 1$.

6 ♣ In the following picture, unit vectors $\mathbf{x}$ are drawn (in black) along with their image $A\mathbf{x}$ (in blue) for a 2×2 matrix A, drawn head to tail. Based on the picture, what vectors appear to be eigenvectors of A. Mark all that apply.



- [x] $[1, 0]^T$
- [x] $[0, 1]^T$
- [ ] $[1, -1]^T$
- [ ] $[1, 1]^T$
- [ ] $[0, 0]^T$
- [ ] $[-1, 2]^T$
- [ ] $[2, 1]^T$

7 ♣ Let A be a 3×3 matrix, and suppose that $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors corresponding to the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 5$ respectively. Which of the following facts must be true? Mark all that apply.

- [ ] The matrix $A - I$ is invertible.
- [x] A is invertible.
- [x] The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
- [ ] The eigenvalues of A^{-1} are -1, -4, and -5.
- [x] The eigenvalues of A^T are 1, 4, and 5.
- [x] There is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
- [ ] $\det(A + 4I) = 0$

8 ♣ If A and B are similar matrices, then A and B share the same (mark all that apply):

- [ ] Eigenspaces
- [x] Eigenvalues
- [x] Determinant
- [ ] Eigenvectors
- [x] Characteristic polynomial

9 ♣ Which of the following vectors are in $\text{Span}([1,1,1]^T, [1,2,3]^T)$:

- ☐ $[2, 3, -3]^T$
- ■ $[-6, -10, -14]^T$
- ■ $[-1, -4, -7]^T$
- ■ $[2, 3, 4]^T$

10 The coordinate vector of the vector $[1,1]^T \in \mathbb{R}^2$ relative to the basis $\mathbf{v_1} = [1,2]^T, \mathbf{v_2} = [3,4]^T$ is

- ☐ $[1/2, -1/2]^T$
- ☐ $[1/2, 1/2]^T$
- ■ $[-1/2, 1/2]^T$
- ☐ $[-1/2, -1/2]^T$

Part II: Short Answer Questions: *Write your answers to the questions below in the space provided.*

11 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ■8 *Administrative Use Only*

a) The eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 0 & -1 & 11 \\ 0 & 0 & -12 \end{bmatrix}$$

are $\lambda =$ 3, $\lambda =$ -1, and $\lambda =$ -12.

b) If A is a 7×7 matrix, and $\lambda = 0$ is an eigenvalue of A with geometric multiplicity 2, then rank$A =$ 5.

c) Give the definition of the rank of a matrix A.

d) Fill in the blank: let A be an $m \times n$ matrix. Then

$$\text{rank}(A) + \underline{\text{nullity}(A)} = n.$$

c) The rank of a matrix A is the dimension of the column space of A (equivalently the dimension of Row A or the number of pivots of A).

☐ **0** ☐ **1** ☐ **2** ☐ **3** ☐ **4** ☐ **5** ☐ **6** ☐ **7** ■ **8** *Administrative Use Only*

a) $\det \begin{bmatrix} 1 & 2 & -3 & 7 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} =$ -24.

b) Give the definition of a basis of a subspace S of $\mathbb{R}^n$.

c) Give four different conditions equivalent to "A is an invertible matrix".

d) If A is a non-invertible square matrix, $\det(A) =$ 0.

e) True or False: 0 cannot be an eigenvalue of any matrix.

f) True or False: If $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then $2\mathbf{x}$ is another eigenvector of A with eigenvalue λ.

b) A basis of S is a set of vectors $\{b_1, \ldots, b_p\}$ such that $\{b_1, \ldots, b_p\}$ is linearly independent and $\text{Span}\{b_1, \ldots, b_p\} = S$.

c) See Theorems 3.12, 3.27, or 4.17 from the book.

Part III: Free Response Questions: *Neatly write complete solutions for these problems directly on the exam paper. Work on scratch paper will not be graded.*

13 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ■8 *Administrative Use Only*

Diagonalize the matrix

$$A = \begin{bmatrix} 1 & -3 & -3 \\ 0 & -5 & -6 \\ 0 & 3 & 4 \end{bmatrix}.$$

In other words, find a diagonal matrix D and an invertible matrix P so that $A = PDP^{-1}$.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -3 & -3 \\ 0 & -5-\lambda & -6 \\ 0 & 3 & 4-\lambda \end{vmatrix} = (1-\lambda)\big((-5-\lambda)(4-\lambda)+18\big)$$

cofactor expansion along column 1.

$$= (1-\lambda)(\lambda^2+\lambda-2) = (1-\lambda)(\lambda+2)(\lambda-1)$$

Eigenvalues are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -2$.

$\underline{\lambda_1 = \lambda_2 = 1}$:

$$A - I = \begin{bmatrix} 0 & -3 & -3 \\ 0 & -6 & -6 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow[\text{Reduce}]{\text{Row}} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} y+z=0 \\ x, z \text{ free} \end{array}$$

$\Rightarrow$ $y = -z$, x, z free

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -z \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow$$

basis for $\lambda_1 = \lambda_2 = 1$ eigenspace $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\underline{\lambda_3 = -2}$:

$$A + 2I = \begin{bmatrix} 3 & -3 & -3 \\ 0 & -3 & -6 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow[\text{Reduce}]{\text{Row}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x+z=0 \\ y+2z=0 \\ z \text{ free} \end{array}$$

$\Rightarrow$ $x = -z$, $y = -2z$, z free

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -2z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow$$

basis for $\lambda_3 = -2$ eigenspace $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

$A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ■8 *Administrative Use Only*

Compute the determinant of the matrix $\begin{bmatrix} 0 & 2 & -4 & 5 \\ 4 & 0 & -4 & 8 \\ 3 & 0 & -5 & 8 \\ -2 & 1 & 3 & -3 \end{bmatrix}$. Be sure to show your work and make clear what steps you are doing.

$$\begin{vmatrix} 0 & 2 & -4 & 5 \\ 4 & 0 & -4 & 8 \\ 3 & 0 & -5 & 8 \\ -2 & 1 & 3 & -3 \end{vmatrix} \underset{\text{swap R1 and R2}}{=} - \begin{vmatrix} 4 & 0 & -4 & 8 \\ 0 & 2 & -4 & 5 \\ 3 & 0 & -5 & 8 \\ -2 & 1 & 3 & -3 \end{vmatrix} \underset{\text{factor 4 from R1}}{=} -4 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 3 & 0 & -5 & 8 \\ -2 & 1 & 3 & -3 \end{vmatrix}$$

$$\underset{\substack{\text{R3}-3\cdot\text{R1} \\ \text{and} \\ \text{R4}+2\cdot\text{R1}}}{=} -4 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 0 & 0 & -2 & 2 \\ 0 & 1 & 1 & 1 \end{vmatrix} \underset{\text{swap R2 and R4}}{=} 4 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & -4 & 5 \end{vmatrix} \underset{\text{R4}-2\cdot\text{R2}}{=} 4 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -6 & 3 \end{vmatrix}$$

$$\underset{\text{factor }-2\text{ from R3}}{=} -8 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -6 & 3 \end{vmatrix} \underset{\text{R4}+6\cdot\text{R3}}{=} -8 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -3 \end{vmatrix} = -8 \cdot (1 \cdot 1 \cdot 1 \cdot (-3)) = 24$$

15 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ■7 *Administrative Use Only*

Compute the eigenvalues and eigenvectors of $\begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix}$. Set $A = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 6 & -\lambda \end{vmatrix} = -\lambda(-1-\lambda) - 6 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

Eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -3$.

$\underline{\lambda_1 = 2:}$ $\quad A - 2I = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \xrightarrow[\text{Reduce}]{\text{Row}} \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} \Rightarrow$ $x - \frac{1}{3}y = 0$, y free

$\Rightarrow$ $x = \frac{1}{3}y$, y free $\quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}y \\ y \end{bmatrix} = y\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ $\Rightarrow$ Basis for $\lambda_1 = 2$ eigenspace $\left\{ \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \right\}$

$\underline{\lambda_2 = -3:}$ $\quad A + 3I = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \xrightarrow[\text{Reduce}]{\text{Row}} \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \Rightarrow$ $x + \frac{1}{2}y = 0$, y free

$\Rightarrow$ $x = -\frac{1}{2}y$, y free $\quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}y \\ y \end{bmatrix} = y\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ $\Rightarrow$ Basis for $\lambda_2 = -3$ eigenspace $\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$

 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ■8 *Administrative Use Only*

Show that all the vectors orthogonal to $\mathbf{u} = [-7, 10, 5]^T \in \mathbb{R}^3$ form a subspace of $\mathbb{R}^3$

Let S be the set of all vectors in $\mathbb{R}^3$ which are orthogonal to $\vec{u}$.
We show that S is a subspace:

We check the 3 subspace criteria:

1) $\vec{0} \cdot \vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ 10 \\ 5 \end{bmatrix} = 0$, thus $\vec{0} \in S$

2) Let $\vec{v}, \vec{w}$ be vectors in S, i.e. $\vec{v} \cdot \vec{u} = 0$ and $\vec{w} \cdot \vec{u} = 0$. Then
$(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u} = 0 + 0 = 0$. Thus $\vec{v} + \vec{w}$ is orthogonal to $\vec{u}$, and $\vec{v} + \vec{w} \in S$.

3) Let $\vec{v}$ be a vector in S, i.e. $\vec{v} \cdot \vec{u} = 0$, and let c be a scalar.
Then $(c\vec{v}) \cdot \vec{u} = c(\vec{v} \cdot \vec{u}) = c \cdot 0 = 0$. Thus $c\vec{v}$ is orthogonal to $\vec{u}$, and $c\vec{v} \in S$.

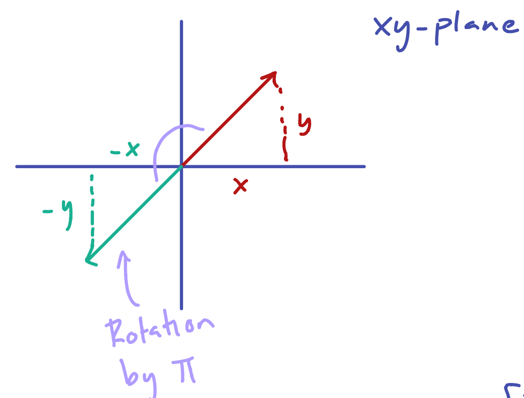
Since S satisfies the subspace criteria, S is a subspace.

17 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ■7 *Administrative Use Only*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that rotates points around the z-axis by an angle of π.
(i) Show that T is a linear transformation, by using the definition of a linear transformation.
(ii) Find the standard matrix of T.

Rotation around the z-axis by an angle of π is given by:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \\ z \end{bmatrix}$$



(i) To show T is linear we check the linearity conditions. Let $\vec{u}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ and c a scalar. Then $\vec{u}_1 + \vec{u}_2 = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}$ and $c \cdot \vec{u}_1 = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}$ and

1. $T(\vec{u}_1 + \vec{u}_2) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} -(x_1+x_2) \\ -(y_1+y_2) \\ (z_1+z_2) \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ -y_1 - y_2 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ -y_2 \\ z_2 \end{bmatrix} = T(\vec{u}_1) + T(\vec{u}_2).$

2. $T(c\vec{u}_1) = T\left(\begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}\right) = \begin{bmatrix} -cx_1 \\ -cy_1 \\ cz_1 \end{bmatrix} = c\begin{bmatrix} -x_1 \\ -y_1 \\ z_1 \end{bmatrix} = cT(\vec{u})$

Since T satisfies $T(\vec{u}_1 + \vec{u}_2) = T(\vec{u}_1) + T(\vec{u}_2)$ and $T(c\vec{u}_1) = cT(\vec{u}_1)$ T is a linear transformation.

(ii)

$$[T] = \left[T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \; T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \; T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$