NS 40123926 597

Math 213
Midterm 2
November 13-15, 2019

KEY

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Instructions

A) Do not write on the barcode area at the top, or near the four circles at the corner of each page.

B) Write your name, section number, and instructor in the space provided, and **COMPLETELY FILL IN** the correct boxes for your BYU ID and for the correct answers in the multiple choice section.

C) Multiple choice questions that are marked with a ♣ may have more than one correct answer. You should mark all correct answers. All other questions have only one correct answer.

D) Multiple choice questions are worth 4 points each.

E) For questions which require a written answer, show all your work in the space provided and justify your answer. Simplify your answers where possible.

F) No books, notes, or calculators are allowed.

G) Do not talk about the test with other students until after the exam period is over.

Part I: Multiple Choice Questions: *(4 points each) Questions marked with a ♣ may have more than one correct answer. Mark* ***all*** *correct answers. The other questions have one correct answer. Choose the best answer for each multiple choice question. Fill in the box completely for each correct answer, but DO NOT leave any marks in the other boxes.*

1 ♣ Let A be a 5×5 matrix, with eigenvalues $\lambda = -1, 2$, and 5. In which of the following situations can we conclude that A is diagonalizable? Mark all that apply, but only mark those options which imply A *must* be diagonalizable.

- [x] The eigenspaces corresponding to the eigenvalues $\lambda = -1$ and $\lambda = 5$ are both 2-dimensional.
- [x] A^T is diagonalizable.
- [x] The eigenvalue $\lambda = 2$ has geometric multiplicity equal to 3.
- [x] $\dim \mathrm{Null}(A + I) = 3$
- [ ] A is invertible.
- [ ] The eigenspaces corresponding to the eigenvalues $\lambda = -1, 2$, and 5 are all 1-dimensional.
- [ ] The eigenvalue $\lambda = 2$ has algebraic multiplicity equal to 3.

2 Let

$$C = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

What is the characteristic polynomial of C?

- [ ] $\lambda^3 + 2\lambda^2 - 3\lambda$
- [ ] $\lambda^2 + 4\lambda - 1$
- [ ] $-\lambda^3 + 2\lambda + 2$
- [x] $-\lambda^3 + 5\lambda^2 - 6\lambda + 2$
- [ ] $2\lambda^3 - \lambda^2 - 3$
- [ ] $-\lambda^3 + 4\lambda^2 + 2\lambda - 1$

3 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects $\mathbf{x}$ over the line $y = -x$. What are the eigenvalues of the standard matrix $[T]$?

- [x] $-1,1$
- [ ] $0,-1$
- [ ] $0,2$
- [ ] $0,1$
- [ ] $1,2$
- [ ] $0,1,-1$
- [ ] $0,1,2$

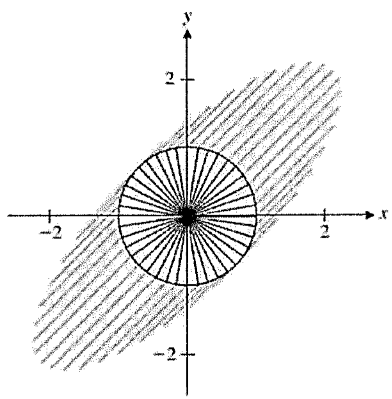
4 ♣ Let A and B be $n \times n$ matrices and k a constant. Which of the following is always correct? Mark all that apply.

- [ ] If $\det(A) = 0$, then $A = 0$
- [x] $\det(AB) = \det(BA)$
- [x] If A is NOT row equivalent to I_n, then $\det(A) = 0$
- [ ] $\det(A + B) = \det(A) + \det(B)$
- [ ] $\det(A^3) = 3\det(A)$
- [x] If $A^2 = 0$, then $\det(A) = 0$
- [x] $\det(kA) = k^n \det(A)$

5 ♣ Suppose B is a 3×3 matrix, and that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of B with eigenvalue $\lambda = 2$, while $\mathbf{w}$ is an eigenvector of B with eigenvalue $\lambda = 3$. Which of the following must be true? Mark all that apply.

- [x] $4 \cdot \mathbf{w}$ is an eigenvector of B with eigenvalue $\lambda = 3$.
- [x] $\mathbf{v}_1 + \mathbf{v}_2$ is an eigenvector of B with eigenvalue $\lambda = 2$.
- [ ] B must have another eigenvalue besides $\lambda = 2$ and $\lambda = 3$.
- [ ] $4 \cdot \mathbf{w}$ is an eigenvector of B with eigenvalue $\lambda = 12$.
- [ ] B is diagonalizable.
- [ ] $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent.

6 ♣ In the following picture, unit vectors $\mathbf{x}$ are drawn (in black) along with their image $A\mathbf{x}$ (in blue) for a 2×2 matrix A, drawn head to tail. Based on the picture, which vectors appear to be eigenvectors of A. Mark all that apply.



- [x] $[1, -1]^T$
- [ ] $[-1, 2]^T$
- [ ] $[2, 1]^T$
- [x] $[-1, -1]^T$
- [ ] $[1, 0]^T$
- [ ] $[0, 0]^T$
- [x] $[1, 1]^T$

7 Compute $\det \begin{bmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

- [x] 16
- [ ] 2
- [ ] 18
- [ ] -10
- [ ] -1
- [ ] 0
- [ ] 15
- [ ] 1

8 The matrix

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

is similar to which of the following diagonal matrices?

- [ ] $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
- [ ] $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$
- [ ] $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- [x] $\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$
- [ ] $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$
- [ ] $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

9 ♣ Which of the following vectors are in Span $\{[1,1,2,-1]^T, [1,2,3,4]^T\}$:

- ☒ $[8,4,12,-28]^T$
- ☒ $[-6,-10,-16,-14]^T$
- ☐ $[4,3,2,1]^T$
- ☐ $[1,-9,4,5]^T$

10 Find the coordinate vector of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathbb{R}^2$ relative to the basis $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$:

- ☐ $\frac{1}{10}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$
- ☐ $\frac{1}{10}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$
- ☒ $\frac{1}{10}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$
- ☐ $\frac{1}{10}\begin{bmatrix} -1 \\ -3 \end{bmatrix}$

Part II: Short Answer Questions: *Neatly write complete solutions for these problems directly on the exam paper. Work on scratch paper will not be graded.*

11 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9 ☐10 ☐11 ☐12 ☐13 ☐14 ☐15

Fill in the blank with the appropriate answer. 2 points per answer.

a) Compute the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}$$

$\lambda^2 - 4\lambda - 7$

b) An eigenvector $\mathbf{x}$ of the matrix

$$B = \begin{bmatrix} -2 & 2 \\ 3 & -1 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = 1$ is given by $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

c) If A is a 3×3 invertible matrix with eigenvalues $\lambda = -2, 4$, and 7, then the eigenvalues of A^{-1} are $\lambda =$ $-1/2$, $\lambda =$ $1/4$, and $\lambda =$ $1/7$.

d) True or false: the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is diagonalizable. FALSE.

e) If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 7$, then $\det \begin{bmatrix} -3a & -3b & -3c \\ d+a & e+b & f+c \\ g & h & i \end{bmatrix} =$ -21.

12 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9 ☐10 ☐11 ☐12 ☐13 ☐14 ☐15

Fill in the blank with the appropriate answer. 2 points per answer.

a) $\det \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} =$ 0.

b) True or False: $\mathbf{0}$ is an eigenvector of every matrix.

c) True or False: If $(A - \lambda I)\mathbf{x} = \mathbf{0}$, and $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}$ is an eigenvector of A.

d) Give the definition of a subspace S of $\mathbb{R}^n$.

A subspace S of $\mathbb{R}^n$ is a set of vectors satisfying:

(1) $\vec{0} \in S$

(2) For all $\vec{u}, \vec{v} \in S$, $\vec{u} + \vec{v} \in S$

(3) For all $\vec{u} \in S$ and all constants c, $c\vec{u} \in S$

e) Give four different conditions equivalent to "A is an invertible matrix".

- 0 is not an eigenvalue of A
- $\det(A) \neq 0$
- $\text{rank}(A) = n$
- $\text{Nul}(A) = \{0\}$
- RREF of A is I_n
- $\text{Col}(A) = \mathbb{R}^n$
- $\text{Row}(A) = \mathbb{R}^n$

(many others)

Part III: Free Response Questions: *Neatly write complete solutions for these problems directly on the exam paper. Work on scratch paper will not be graded.*

13 $\square$ 0 $\square$ 1 $\square$ 2 $\square$ 3 $\square$ 4 $\square$ 5 $\square$ 6 *Administrative Use Only*

Compute the determinant of the matrix $\begin{bmatrix} -2 & 2 & 4 & 6 \\ 2 & -2 & -6 & 4 \\ -3 & 6 & -4 & 1 \\ 3 & -6 & 4 & 2 \end{bmatrix}$.

Be sure to show your work and make it clear what steps you are doing.

$$\begin{vmatrix} -2 & 2 & 4 & 6 \\ 2 & -2 & -6 & 4 \\ -3 & 6 & -4 & 1 \\ 3 & -6 & 4 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 4 & 6 \\ 0 & 0 & -2 & 10 \\ -3 & 6 & -4 & 1 \\ 0 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 4 & 6 \\ 0 & 0 & -2 & 10 \\ 0 & 3 & -10 & -8 \\ 0 & 0 & 0 & 3 \end{vmatrix}$$

replace R_2 w/ R_1+R_2
replace R_4 w/ R_3+R_4

Replace R_3 w/ $-\frac{3}{2}R_1+R_3$

$R_2 \Leftrightarrow R_3$

$$= - \begin{vmatrix} -2 & 2 & 4 & 6 \\ 0 & 3 & -10 & -8 \\ 0 & 0 & -2 & 10 \\ 0 & 0 & 0 & 3 \end{vmatrix} = -\left((-2)\cdot 3\cdot(-2)\cdot 3\right) = \boxed{-36}$$

0 1 2 3 4 5 6 *Administrative Use Only*

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2y - 7z \\ -3x + \pi y \\ x \\ x + y - z \end{bmatrix}.$$

Prove that T is a linear transformation.

(1) $$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} 2(y_1+y_2) - 7(z_1+z_2) \\ -3(x_1+x_2) + \pi(y_1+y_2) \\ x_1+x_2 \\ (x_1+x_2) + (y_1+y_2) - (z_1+z_2) \end{bmatrix}$$

$$= \begin{bmatrix} (2y_1 - 7z_1) + (2y_2 - 7z_2) \\ (-3x_1 + \pi y_1) + (-3x_2 + \pi y_2) \\ x_1 + x_2 \\ (x_1+y_1-z_1) + (x_2+y_2-z_2) \end{bmatrix} = \begin{bmatrix} 2y_1 - 7z_1 \\ -3x_1 + \pi y_1 \\ x_1 \\ x_1+y_1-z_1 \end{bmatrix} + \begin{bmatrix} 2y_2 - 7z_2 \\ -3x_2 + \pi y_2 \\ x_2 \\ x_2+y_2-z_2 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$$

(2) $$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} 2(cy) - 7(cz) \\ -3(cx) + \pi(cy) \\ cx \\ cx + cy - cz \end{bmatrix} = \begin{bmatrix} c(2y - 7z) \\ c(-3x + \pi y) \\ cx \\ c(x+y-z) \end{bmatrix}$$

$$= c\begin{bmatrix} 2y - 7z \\ -3x + \pi y \\ x \\ x+y-z \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right).$$

Therefore T is a linear transformation.

15 $\square$0 $\square$1 $\square$2 $\square$3 $\square$4 $\square$5 $\square$6 $\square$7 $\square$8 $\square$9 *Administrative Use Only*

Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

In other words, find a diagonal matrix D and an invertible matrix P so that we have $A = PDP^{-1}$.

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)^2-1\right) = (1-\lambda)\left(1-2\lambda+\lambda^2-1\right)$$

$$= (1-\lambda)\lambda(\lambda-2) = 0$$

$\lambda = 0, 1, 2$

$\lambda = 0$: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$ eigenvector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\lambda = 1$: $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$ eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 2$: $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$ eigenvector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

16 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9 *Administrative Use Only*

Let A be an $m \times n$ matrix.
(i) Define the null space of A.

$Nul(A) = \{ \vec{x} \in \mathbb{R}^n$ with $A\vec{x} = \vec{0} \}$.

(ii) Prove that the null space is a subspace of $\mathbb{R}^n$.

(1) $A\vec{0} = \vec{0} \longrightarrow \vec{0} \in Nul(A)$

(2) If $\vec{x}, \vec{y} \in Nul(A)$, $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$,
thus $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0} \implies \vec{x}+\vec{y} \in Nul(A)$

(3) If $\vec{x} \in Nul(A)$, $A\vec{x} = \vec{0}$,
thus for any scalar c,
$A(c\vec{x}) = c(A\vec{x}) = c\vec{0} = \vec{0} \implies c\vec{x} \in Nul(A)$

Therefore $Nul(A)$ is a subspace of $\mathbb{R}^n$.