Math 213 - Winter 2020
Exam 2—KEY
March 2020

Time: 4 hours

Instructions:

- Complete all exam questions from this sheet, and enter your answers in Learning Suite, in the space provided under the exam "Midterm 2".
- This exam is closed book.
- Calculators, notes, books, online resources, and help from others is not allowed.
- Please do not communicate with others about this exam until after the exam period has closed.

Part I: Multiple Choice Questions: *(4 points each) Questions marked with a ♣ may have more than one correct answer. Enter **all** correct answers into the space provided in Learning Suite. The other questions have one correct answer. Choose the best answer for each multiple choice question.* ***Correct answers underlined and in blue.***

1. ♣ Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Which of the following statements about T must be true? Mark all that apply.

   A. **$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.**

   B. If $T(\mathbf{u}) = \mathbf{0}$ then $\mathbf{u} = \mathbf{0}$.

   C. **The domain of T is $\mathbb{R}^n$.**

   D. **$T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in $\mathbb{R}^n$ and all c in $\mathbb{R}$.**

   E. $T(\mathbf{u} \cdot \mathbf{v}) = T(\mathbf{u}) \cdot T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.

   F. The range of T is $\mathbb{R}^m$.

2. ♣ Which of the following sets are subspaces of $\mathbb{R}^3$? Mark all that apply.

   A. **The plane defined by the equation $3x + 2y - z = 0$.**

   B. Null A, where A is a 3×2 matrix.

   C. **Col A^T, where A is a 5×3 matrix.**

   D. The set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $x + y + z < 1$.

   E. The set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $x = y$ and $z = y - 1$.

   F. The line defined by the vector form equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$.

3. Let $\mathcal{B}$ be the basis for $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Find the coordinator vector of $\mathbf{x} = \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}$ with respect to the basis $\mathcal{B}$.

A. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$

B. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}.$

C. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$

D. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}.$

E. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}.$

F. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}.$

4. Let S be the set consisting of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $x \geq y$. Consider the following statements:

I. S contains the zero vector.
II. S is closed under vector addition.
III. S is closed under scalar multiplication.

Which of the following statements is true?

A. Statements I., II., and III. are true, and hence S is a subspace.

B. Statements I. and III. are true, while II. is false.

C. Statements II. and III. are true, while I. is false.

D. **Statements I. and II. are true, while III. is false.**

E. Statement I. is true, while II. and III. are false.

F. Statement II. is true, while I. and III. are false.

G. Statement III. is true, while I. and II. are false.

H. Statement I., II., and III. are all false.

5. ♣ Let A be a 9×6 matrix, with rank $A = 6$. Which of the following statements must be true? Mark all that apply.

   A. **The columns of A are linearly independent.**

   B. $\operatorname{rank} A + \dim(\operatorname{Nul} A) = 6$

   C. The columns of A span $\mathbb{R}^9$.

   D. Nullity $A \geq 3$

   E. **The rows of A are linearly dependent.**

   F. $\operatorname{Nul} A = \{\mathbf{0}\}$

   G. $\dim(\operatorname{Row} A) = 9$

6. ♣ Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a subspace S of $\mathbb{R}^5$. Which of the following statements must be true? Mark all that apply.

   A. $\mathcal{B}$ is the unique basis for S.

   B. **The set $\{\mathbf{b}_1, \mathbf{b}_2\}$ is linearly independent.**

   C. **The set $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 + \mathbf{b}_2\}$ is also a basis for S.**

   D. **The set $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{0}\}$ spans S.**

   E. The set $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{0}\}$ is also a basis for S.

   F. $\dim S = 3$

7. ♣ Let A be an $n \times n$ matrix. Which of the following must be true? Mark all that apply.

A. **If $\operatorname{rank}(A) = n$, $\det(A) \neq 0$.**

B. If $\operatorname{nullity}(A) = 0$, $\det(A) = 0$.

C. **If the rows of A are linearly dependent, $\det(A) = 0$.**

D. If $\det(A) = 0$, $A\mathbf{x} = \mathbf{0}$ has a unique solution.

E. **If $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^{\mathbf{n}}$, then $\det(A) = 0$.**

F. **If $\det(A) \neq 0$, $\operatorname{col}(A) = \mathbb{R}^n$.**

8. If A and B are $n \times n$ matrices with $\det(A) = 3$ and $\det(B) = -2$, what is $\det(A^{-1}B^2A^T)$?

A. 4

B. -4

C. 3

D. -3

E. 0

F. -2

G. $\frac{4}{9}$

H. Not enough information to determine.

9. ♣ Let $\mathbf{x}$ and $\mathbf{y}$ be linearly independent eigenvectors of the $n \times n$ matrix A, corresponding to the same eigenvalue eigenvalue λ. Let I_n be the $n \times n$ identity matrix. Which of the following must be true? Mark all that apply.

A. **$\mathbf{x} \neq \mathbf{0}$.**

B. **$\mathbf{x}, \mathbf{y} \in \operatorname{Nul}(A - \lambda I_n)$.**

C. **$\mathbf{x} + \mathbf{y}$ is also an eigenvector of A with eigenvalue λ.**

D. **If A is invertible, $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1} corresponding to the same eigenvectors.**

E. $A - \lambda I_n$ is invertible.

F. **$A^k\mathbf{x} = \lambda^k\mathbf{x}$.**

10. ♣ Suppose A is similar to B. Which of the following must be true? Mark all that apply.

   A. **$AP = PB$ for some invertible matrix P.**

   B. $4A$ is similar to B.

   C. **A^3 is similar to B^3.**

   D. $\det(A) = \det(B)$

   E. **A and B have the same characteristic polynomial.**

   F. A and B have the same eigenvectors.

11. ♣ Suppose A is a 5×5 matrix with three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Which of the following would guarantee that A is diagonalizable? Mark all that apply.

   A. **λ_1 has geometric multiplicity 3.**

   B. **λ_1 and λ_2 both have geometric multiplicity 2.**

   C. $\lambda_1, \lambda_2, \lambda_3$ each have geometric multiplicity 1.

   D. λ_1 has algebraic multiplicity 3.

   E. λ_1 and λ_2 both have algebraic multiplicity 2.

   F. **The algebraic and geometric multiplicities are equal for each eigenvalue of A.**

Part II: Short Answer Questions: *Enter your answers into the space provided in Learning Suite.*

12. **True** or False: Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix A.

13. **True** or False: If S is a 3-dimensional subspace of $\mathbb{R}^5$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a linearly independent set of vectors in S, then $\mathcal{B}$ is a basis for S.

14. True or **False**: If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly dependent set of vectors that spans a subspace H, then $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ will also span H.

15. True of **False**: $\det(A + B) = \det(A) + \det(B)$ for all $n \times n$ matrices A and B.

16. **True** of False: If A is an 3×3 matrix, then $\det(-A) = -\det(A)$.

17. Let W be the subspace of $\mathbb{R}^3$ given by

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \right\}.$$

Find a basis for W, and $\dim W$.

Solution. *Basis for W:* $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$, $\dim W = 2$

18. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$, and let $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ be the coordinate vector of a vector $\mathbf{x}$ with respect to $\mathcal{B}$. Find the vector $\mathbf{x}$.

Solution. $\mathbf{x} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

19. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ -3 & 2 & -1 \end{bmatrix}.$$

Find bases for Row A, Col A, and Null A. (Input the vectors in each basis as column vectors.)

Solution. *RREF of A is* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Basis for Row A: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. *Basis for* Col A: $\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$. *Basis for* Null A: $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

20. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates vectors 90° counterclockwise about the origin, and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ 4x + y \end{bmatrix}.$$

Find the standard matrices for the linear transformations S, T, and $S \circ T(\mathbf{x}) = S(T(\mathbf{x}))$.

Solution. $[S] = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$

$[T] = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$[S \circ T] = [S][T] = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 1 & -4 \end{bmatrix}$

21. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 1 & 2 & -2 \\ 5 & 1 & 3 & 2 \end{bmatrix}$$

Solution.

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 4 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 1 & 2 & -2 \\ 5 & 1 & 3 & 2 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 4 & 1 \\ 0 & -1 & -8 & -1 \\ 0 & 2 & 6 & -1 \\ 0 & -4 & -17 & -3 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 4 & 1 \\ 0 & -1 & -8 & -1 \\ 0 & 0 & -10 & -3 \\ 0 & 0 & 15 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 4 & 1 \\ 0 & -1 & -8 & -1 \\ 0 & 0 & -10 & -3 \\ 0 & 0 & 0 & -7/2 \end{vmatrix} \\ &= 1(-1)(-10)(-7/2) \\ &= \underline{\mathbf{-35}} \end{aligned}$$

22. Diagonalize the matrix

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

In other words, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution. *Characteric Polynomial:* $-\lambda^3 - 2\lambda^2 - \lambda = -\lambda(\lambda + 1)^2$

Eigenvalues: $\lambda = 0, -1, -1$

Eigenvector for $\lambda = 0$*:* $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Eigenvectors for $\lambda = -1$*:* $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$