NS 40121820 **001**

Math 213
Final Exam
December, 2019

Name:____________________

Section:____________________

Instructor:____________________

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Instructions

A) Do not write on the barcode area at the top of each page, or near the four circles on each page.

B) Write your name, section number, and instructor in the space provided, and **COMPLETELY FILL IN** the correct boxes for your BYU ID and for the correct answers in the multiple choice section.

C) Multiple choice questions that are marked with a ♣ may have more than one correct answer. You should mark all correct answers. All other questions have only one correct answer.

D) Multiple choice questions are worth 3 points each. For multiple choice questions with more than one correct answer, each option will be graded with equal weight.

E) For questions which require a written answer, show all your work in the space provided and justify your answer. Simplify your answers where possible.

F) No books, notes, or calculators are allowed.

G) Do not talk about the test with other students until after the exam period is over.

Part I: Multiple Choice Questions: *(3 points each) Questions marked with a ♣ may have more than one correct answer. Mark* ***all*** *correct answers. The other questions have one correct answer. Choose the best answer for each multiple choice question. Fill in the box completely for each correct answer, but DO NOT leave any marks in the other boxes.*

1 ♣ Suppose that A is a 7×5 matrix with linearly independent columns. Which of the following statements must be true? Mark all that apply.

- ☐ The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^7$.
- ☐ $\text{rank}(A) = 5$
- ☐ A has 5 pivots.
- ☐ The columns of A span $\mathbb{R}^7$.
- ☐ The system represented by the augmented matrix $[\,A\,|\,\mathbf{b}]$ is consistent for all $\mathbf{b} \in \mathbb{R}^7$.
- ☐ The equation $A\mathbf{x} = \mathbf{0}$ has a unique solution.
- ☐ $\text{Nullity}(A) = 2$

2 ♣ Suppose that U is a $n \times n$ matrix with orthonormal columns. Which of the following must be true? Mark all that apply.

- ☐ $\text{Nullity}(A) = n$
- ☐ U has n nonzero singular values.
- ☐ U is orthogonally diagonalizable.
- ☐ U is invertible, with $U^{-1} = U^T$.
- ☐ U has an eigenvector $\mathbf{v}$ with eigenvalue $\lambda = 1$.
- ☐ $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.
- ☐ $\text{rank}(A) = n$
- ☐ U has orthonormal rows.

3 Find the singular values of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}.$$

- ☐ $1, -1$
- ☐ $4, 2$
- ☐ $2, \sqrt{2}, 0$
- ☐ $1, -1, 0$
- ☐ $2, \sqrt{2}$
- ☐ $4, 2, 0$

4 ♣ Let A be a 4×7 matrix with $\text{rank}(A) = 3$. Which of the following must be true? Mark all that apply.

- ☐ $\dim(\text{Null}(A)^{\perp}) = 4$
- ☐ 0 is a singular value of A.
- ☐ $\dim(\text{Row}(A)^{\perp}) = 4$
- ☐ $\dim(\text{Row}(A)) = 4$
- ☐ $\dim(\text{Null}(A)) = 1$
- ☐ $\dim(\text{Col}(A)) = 3$

5 Let A be a 5×7 matrix with rank 4. What is the dimension of the null space of A?

- ☐ 3
- ☐ 4
- ☐ 5
- ☐ 1
- ☐ 0
- ☐ 2

6 Let

$$C = \begin{bmatrix} 1 & 1 \\ x & x \end{bmatrix}$$

Find $x \in \mathbb{R}$ such that $C^2 = \mathbf{0}$:

- ☐ -1
- ☐ -2
- ☐ 0
- ☐ 1
- ☐ 2
- ☐ -3

7 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates $\mathbf{x}$ counterclockwise by $\pi/2$ radians. What is the standard matrix $[T]$?

- ☐ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- ☐ $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
- ☐ $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

8 Let A, B, and C be $n \times n$ matrices. The property $A(BC) = (AB)C$ is called.

- ☐ commutative
- ☐ additive identity
- ☐ associative
- ☐ multiplicative identity
- ☐ left distributivity
- ☐ right distributive

9 Solve

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 2 & -14 \\ 6 & -28 \end{bmatrix}$$

for $a, b \in \mathbb{R}$:

- ☐ $a = -2, b = -7$.
- ☐ $a = 7, b = -2$.
- ☐ $a = -7, b = -2$.
- ☐ $a = 2, b = 7$.
- ☐ $a = -7, b = 2$.
- ☐ $a = -2, b = 7$.
- ☐ $a = 2, b = -7$.

10 ♣ Let A be an invertible square matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$. Which of the following must be true?

- ☐ $A\mathbf{x} = \lambda\mathbf{x}$
- ☐ $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$
- ☐ $\det(A - \lambda I) = 0$
- ☐ $\mathbf{x} = \mathbf{0}$
- ☐ $A - \lambda I$ is invertible

11 ♣ Which of the following matrices are orthogonally diagonalizable? Mark all that apply.

- ☐ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
- ☐ $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 4 & 0 \\ -1 & 0 & -3 \end{bmatrix}$

12 Compute the characteristic polynomial of $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$.

- ☐ $-\lambda^3 + 4\lambda^2 - 5\lambda$
- ☐ $-\lambda^3 + 3\lambda^2 - 3\lambda + 1$
- ☐ $-\lambda^3 + 4\lambda^2 + 3\lambda$
- ☐ $-\lambda^3 - 3\lambda^2 + 5\lambda + 2$
- ☐ $-\lambda^3 - 3\lambda^2 - 5\lambda - 1$
- ☐ $-\lambda^3 - 4\lambda^2 - 5\lambda - 2$
- ☐ $-\lambda^3 + 4\lambda^2 - 5\lambda + 2$
- ☐ None of the above

13 ♣ Let A be a 3×3 real matrix with $A^T = A$ and eigenvalues $2, 2, -1$ with eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ respectively. Which of the following must be true? Mark all that apply.

- ☐ A is diagonal.
- ☐ $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$
- ☐ A is invertible.
- ☐ A is orthogonal.
- ☐ There is an orthogonal Q such that $Q^T AQ$ is diagonal.
- ☐ $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0$
- ☐ The geometric multiplicity of $\lambda = 2$ is two.

Part II: Short Answer Questions: *Write your answers to the questions below in the space provided.*

14 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9 ☐10 ☐11 ☐12 *Administrative Use On*

a) Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{v}_1 = \begin{bmatrix}1 & 1 & 0 & -1\end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix}0 & 0 & 1 & 4\end{bmatrix}^T$. Find a basis for $W^\perp$.

$$\left\{ \qquad\qquad \right\}$$

b) Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{v}_1 = \begin{bmatrix}2 & 1 & 0 & 1\end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix}0 & -1 & 1 & 1\end{bmatrix}^T$, and let $\mathbf{y} = \begin{bmatrix}1 & 2 & 1 & 2\end{bmatrix}^T$. Find the projection of $\mathbf{y}$ onto W and the perpendicular component of $\mathbf{y}$ to W.

$$\text{proj}_W\mathbf{y} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \quad \text{and} \quad \text{perp}_W\mathbf{y} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

c) State the precise definition of the following: The vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ span a subspace W if

d) If A is a 3×3 matrix with $\det A = -3$, then $\det(2A) =$__________.

e) If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6$, then $\det \begin{bmatrix} d & e & f \\ 3a & 3b & 3c \\ g & h & i \end{bmatrix} =$__________.

15 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 *Administrative Use Only*

a) Suppose that A is a 3×3 invertible matrix that is upper triangular and whose entry a_{11} in the first row and first column is equal to -1. What is the entry in the first row and first column of A^{-1}? ______________

b) State the precise definition of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

c) State the precise definition of an eigenvalue for a matrix A.

d) If A is a 3×3 matrix with eigenvalues $1, 0, -1$ and corresponding eigenvectors $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ respectively, and if $\mathbf{x} = -2\mathbf{u} - \mathbf{v} + 3\mathbf{w}$, then

$$A^{1001}\mathbf{x} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}.$$

e) Write the Spectral Decomposition of the matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$:

$A =$ ____

Part III: Free Response Questions: *Neatly write complete solutions for these problems directly on the exam paper. Work on scratch paper will not be graded.*

16 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 *Administrative Use Only*

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

17 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 *Administrative Use Only*

Let

$$B = \begin{bmatrix} 1 & 1 & 3 & 1 & -2 \\ -3 & 1 & -5 & 0 & -4 \\ -1 & 1 & -1 & 1 & -4 \end{bmatrix}.$$

Find bases for each of the following subspaces: Row(B), Col(B), and Null(B). Clearly label which subspace each basis belongs to.

18 □0 □1 □2 □3 □4 □5 □6 □7 □8 *Administrative Use Only*

Orthogonally diagonalize the following matrix

$$A = \begin{bmatrix} 3 & -3 \\ -3 & -5 \end{bmatrix}$$

(i.e. find a diagonal matrix D and an orthogonal matrix Q such that $A = QDQ^T$).

19 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 *Administrative Use Only*

Find a QR-decomposition of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

20 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 *Administrative Use Only*

Find a singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}.$$