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Math 213
Practice Final
December, 2019

Name: Key

Section:

Instructor:

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Instructions

A) Do not write on the barcode area at the top of each page, or near the four circles on each page.

B) Write your name, section number, and instructor in the space provided, and **COMPLETELY FILL IN** the correct boxes for your BYU ID and for the correct answers in the multiple choice section.

C) Multiple choice questions that are marked with a ♣ may have more than one correct answer. You should mark all correct answers. All other questions have only one correct answer.

D) Multiple choice questions are worth 3 points each. For multiple choice questions with more than one correct answer, each option will be graded with equal weight.

E) For questions which require a written answer, show all your work in the space provided and justify your answer. Simplify your answers where possible.

F) No books, notes, or calculators are allowed.

G) Do not talk about the test with other students until after the exam period is over.

Part I: Multiple Choice Questions: *(3 points each) Questions marked with a ♣ may have more than one correct answer. Mark **all** correct answers. The other questions have one correct answer. Choose the best answer for each multiple choice question. Fill in the box completely for each correct answer, but DO NOT leave any marks in the other boxes.*

1 ♣ Suppose A is a 3×4 matrix with a pivot in each row. Which of the following must be true? Mark all that apply.

■ The equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for each $\mathbf{b} \in \mathbb{R}^3$.

□ The columns of A are linearly independent.

□ The equation $A\mathbf{x} = \mathbf{0}$ has a unique solution.

□ There is a vector $\mathbf{b} \in \mathbb{R}^3$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

■ $\text{Nullity}(A) = 1$

■ The columns of A span $\mathbb{R}^3$.

□ $\text{rank}(A) = 4$

2 ♣ Suppose that U is a $n \times n$ orthogonal matrix. Which of the following must be true? Mark all that apply.

□ U has n distinct eigenvalues.

■ The columns of U are linearly independent.

□ Every eigenvector of U has length 1.

■ $\text{rank}(A) = n$

□ $\det U = 1$

■ $U^T U$ is the identity matrix.

■ $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

■ U has orthonormal columns.

3 Find the singular values of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

- [ ] $5, 3, 1$
- [ ] $\sqrt{7}, \sqrt{2}$
- [x] $\sqrt{5}, \sqrt{3}, 1$
- [ ] $1, -1, 0$
- [ ] $7, 2, 0$
- [ ] $\sqrt{7}, \sqrt{2}, 0$

4 Performing the Gram-Schmidt procedure on the vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

produces which of the following sets? (Only select one of the following choices.)

- [ ] $\left\{ \begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}^T, \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}^T \right\}$
- [ ] $\left\{ \begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix}^T, \begin{bmatrix} -\frac{1}{3} & 0 & -\frac{1}{3} & 1 \end{bmatrix}^T, \begin{bmatrix} -\frac{22}{21} & \frac{20}{21} & -\frac{2}{21} & \frac{2}{7} \end{bmatrix}^T \right\}$
- [ ] $\left\{ \begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix}^T, \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} & 1 \end{bmatrix}^T \right\}$
- [ ] $\left\{ \begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}^T, \begin{bmatrix} -\frac{1}{3} & 0 & -\frac{1}{3} & 1 \end{bmatrix}^T \right\}$
- [x] $\left\{ \begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}^T, \begin{bmatrix} -\frac{22}{21} & \frac{20}{21} & -\frac{2}{21} & \frac{2}{7} \end{bmatrix}^T \right\}$
- [ ] $\left\{ \begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix}^T, \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} -\frac{22}{21} & \frac{20}{21} & -\frac{2}{21} & \frac{2}{7} \end{bmatrix}^T \right\}$

5 ♣ Let A be a 5×17 matrix with rank 4. What is the dimension of the null space of A:

- ☐ 3
- ☐ 4
- ■ greater than 4
- ☐ 1
- ☐ 0
- ☐ 2

6 Let

$$C = \begin{bmatrix} 1 & 1 & 1 \\ x & x & x \\ 0 & 0 & 0 \end{bmatrix}$$

Find $x \in \mathbb{R}$ such that $C^2 = \mathbf{0}$:

- ■ -1
- ☐ -2
- ☐ 0
- ☐ 1
- ☐ 2
- ☐ -3

7 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects $\mathbf{x}$ across the y-axis. What is the standard matrix $[T]$?

- ☐ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- ☐ $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
- ■ $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

8 Let A, B, and C be $n \times n$ matrices. The property $A(B + C) = AB + AC$ is called.

- ☐ commutative
- ☐ additive identity
- ☐ associative
- ☐ multiplicative identity
- ■ left distributive
- ☐ right distributive

9 ♣ If $A^{-1} = A^T$, then the matrix A is always (mark all that apply):

- ☐ the zero matrix
- ☐ the identity matrix
- ■ orthogonal
- ☐ symmetric
- ☐ triangular
- ■ square
- ☐ diagonal
- ■ invertible

10 Let $\mathbf{u}_1 = [1,1]^T, \mathbf{u}_2 = [1,-1]^T, \mathbf{y} = [3,5]^T$. Find the coordinates of $\mathbf{y}$ relative to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$.

- ☐ $[-1,-4]^T$
- ☐ $[-1,4]^T$
- ■ $[4,-1]^T$
- ☐ $[1,4]^T$
- ☐ $[1,-4]^T$
- ☐ $[4,1]^T$
- ☐ $[-4,-1]^T$
- ☐ $[-4,1]^T$

11 ♣ Let A be a square matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$. Which of the following must be true?

- [ ] $2\mathbf{x}$ is and eigenvector of A with eigenvalue 2λ
- [x] λ is an eigenvalue of A^T
- [ ] $\lambda \neq 0$
- [x] $A^3\mathbf{x} = \lambda^3\mathbf{x}$
- [x] $\mathbf{x} \in \mathrm{Nul}(A - \lambda I)$

12 ♣ Which of the following matrices are orthogonally diagonalizable? Mark all that apply.

- [x] $\begin{bmatrix} 1 & -2 & 4 \\ -2 & 4 & 5 \\ 4 & 5 & 3 \end{bmatrix}$
- [ ] $\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -6 \\ 0 & 0 & 5 \end{bmatrix}$
- [ ] $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$
- [x] $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$
- [x] $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
- [ ] $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 5 & 1 \\ -1 & 1 & -3 \end{bmatrix}$

13 Compute the characteristic polynomial of $\begin{bmatrix} 3 & -2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

- ■ $-\lambda^3 + 3\lambda^2 - 7\lambda + 5$
- □ $-\lambda^3 + 3\lambda^2 - 7\lambda$
- □ $-\lambda^3 + 4\lambda^2 - 5\lambda + 4$
- □ $-\lambda^3 - 3\lambda^2 - 7\lambda - 5$
- □ $-\lambda^3 + 4\lambda^2 - 5\lambda$
- □ $-\lambda^3 - 2\lambda^2 - 6\lambda + 2$
- □ $-\lambda^3 + 2\lambda^2 - 6\lambda + 2$
- □ None of the above

14 ♣ Suppose $Q^T AQ = D$ where $Q^T Q = I$ and D is diagonal. Which of the following must be true? Mark all that apply.

- □ A has an eigenvalue that is an imaginary number
- ■ $A^T = A$
- □ A has all distinct eigenvalues
- ■ There is an orthonormal basis of $\mathbb{R}^n$ consisting of eignevectors of A
- ■ The columns of Q are an orthonormal set
- ■ The eigenvalues of A are the eigenvalues of D

Part II: Short Answer Questions: *Write your answers to the questions below in the space provided.*

15 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ■7 *Administrative Use Only*

a) Let W be the subspace of $\mathbb{R}^5$ spanned by the vectors $\mathbf{v}_1 = \begin{bmatrix} 3 & 2 & -1 & 3 & -3 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 3 & 2 & 2 & -3 & 3 \end{bmatrix}^T$. Find a basis for $W^{\perp}$.

$$\left\{ \begin{bmatrix} -2/3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

b) Let W be the subspace of $\mathbb{R}^5$ spanned by the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T$, and let $\mathbf{y} = \begin{bmatrix} -1 & 2 & 1 & 0 \end{bmatrix}^T$. Find the projection of $\mathbf{y}$ onto W and the perpendicular component of $\mathbf{y}$ to W.

$$\text{proj}_W\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \text{perp}_W\mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

c) State the precise definition of the following: The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthonormal set if

$\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$, and $\|\vec{v}_i\| = 1$ for all $1 \leq i \leq p$.

d) Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

What is A^{10}?

$$A^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ■ 7 *Administrative Use Only*

a) Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Find the inverse of A.

$$A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$$

b) Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

c) Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

d) Give the precise definition of a subspace S of $\mathbb{R}^n$:

A subspace S of $\mathbb{R}^n$ is a set of vectors in $\mathbb{R}^n$ such that

(1) $\vec{0}$ is in S.

(2) If $\vec{u}$ and $\vec{v}$ are in S then $\vec{u} + \vec{v}$ is in S.

(3) If $\vec{u}$ is in S and c is a scalar, then $c\vec{u}$ is in S.

e) True or false: every square matrix is the product of elementary matrices?

False. Only invertible matrices can be written as products of elementary matrices.

17 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ■7 *Administrative Use Only*

a) If $\det A = 3$ and $\det B = -4$, then $\det(A^{-1}B^T) =$ $-4/3$.

b) $\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} =$ 0.

c) If A has eigenvalues λ_1, λ_2 with eigenvectors $\mathbf{u}, \mathbf{v}$ respectively, then

$$A^k(c_1\mathbf{u} + c_2\mathbf{v}) = c_1 \lambda_1^k \vec{u} + c_2 \lambda_2^k \vec{v}.$$

d) Write the Spectral Decomposition of the matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$:

$$A = 3 \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} - 2 \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$= 3 \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} - 2 \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}$$

Part III: Free Response Questions: *Neatly write complete solutions for these problems directly on the exam paper. Work on scratch paper will not be graded.*

18 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ■7 *Administrative Use Only*

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -4 & -12 \\ 1 & -1 & -3 \\ -3 & 7 & 22 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 2 & -4 & -12 & 1 & 0 & 0 \\ 1 & -1 & -3 & 0 & 1 & 0 \\ -3 & 7 & 22 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -3 & 0 & 1 & 0 \\ 2 & -4 & -12 & 1 & 0 & 0 \\ -3 & 7 & 22 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -3 & 0 & 1 & 0 \\ 0 & -2 & -6 & 1 & -2 & 0 \\ 0 & 4 & 13 & 0 & 3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -3 & 0 & 1 & 0 \\ 0 & -2 & -6 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 6 & -2 & 3 \\ 0 & -2 & 0 & 13 & -8 & 6 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 6 & -2 & 3 \\ 0 & 1 & 0 & -13/2 & 4 & -3 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 2 & 0 \\ 0 & 1 & 0 & -13/2 & 4 & -3 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$\Longrightarrow \quad A^{-1} = \begin{bmatrix} -1/2 & 2 & 0 \\ -13/2 & 4 & -3 \\ 2 & -1 & 1 \end{bmatrix}$$

19 | ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ■7 *Administrative Use Only*

Let

$$B = \begin{bmatrix} -2 & -2 & -2 & 1 & -3 \\ 2 & 2 & 1 & 0 & 4 \\ -6 & -6 & -3 & 1 & -9 \\ 12 & 12 & 6 & -2 & 18 \end{bmatrix}.$$

Find bases for each of the following subspaces: Row(B), Col(B), and Null(B). Clearly label which subspace each basis belongs to.

$$\begin{bmatrix} -2 & -2 & -2 & 1 & -3 \\ 2 & 2 & 1 & 0 & 4 \\ -6 & -6 & -3 & 1 & -9 \\ 12 & 12 & 6 & -2 & 18 \end{bmatrix} \to \begin{bmatrix} -2 & -2 & -2 & 1 & -3 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & -6 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} -2 & -2 & -2 & 1 & -3 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} -2 & -2 & -2 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -2 & -2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 + x_5 &= 0 \\ x_3 + 2x_5 &= 0 \\ x_4 + 3x_5 &= 0 \end{aligned} \implies \begin{aligned} x_1 &= -x_2 - x_5 \\ x_3 &= -2x_5 \\ x_4 &= -3x_5 \end{aligned} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 - x_5 \\ x_2 \\ -2x_5 \\ -3x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -2 \\ -3 \\ 1 \end{bmatrix}$$

Null A basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ -3 \\ 1 \end{bmatrix} \right\}$,

Col A basis $\left\{ \begin{bmatrix} -2 \\ 2 \\ -6 \\ 12 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$

Row A basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ■7 *Administrative Use Only*

Orthogonally diagonalize the following matrix

$$A = \begin{bmatrix} -5 & -2 & -1 \\ -2 & -2 & 2 \\ -1 & 2 & -5 \end{bmatrix}$$

(i.e. find a diagonal matrix D and an orthogonal matrix Q such that $A = QDQ^T$).

$$\det(A-\lambda I) = \begin{vmatrix} -5-\lambda & -2 & -1 \\ -2 & -2-\lambda & 2 \\ -1 & 2 & -5-\lambda \end{vmatrix} = (-5-\lambda)\begin{vmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 2 \\ -1 & -5-\lambda \end{vmatrix} - 1\begin{vmatrix} -2 & -2-\lambda \\ -1 & 2 \end{vmatrix}$$

$$= (-5-\lambda)\big((-2-\lambda)(-5-\lambda)-4\big) + 2\big(-2(-5-\lambda)+2\big) - \big(-4+(-2-\lambda)\big) = -\lambda^3 - 12\lambda^2 - 36\lambda$$

$$= -\lambda(\lambda+6)(\lambda+6) \quad \Longrightarrow \text{Eigenvalues } \lambda_1 = -6,\ \lambda_2 = -6,\ \lambda_3 = 0$$

$\underline{\lambda_1 = \lambda_2 = -6:}$

$$A + 6I = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{array}{l} x = 2y+z \\ y, z \text{ free} \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y+z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow \vec{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \longleftarrow$$

Not orthogonal, must perform Gram-Schmidt to make orthogonal.

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 1/5 \\ -2/5 \\ 1 \end{bmatrix} \xrightarrow[\text{Normalize}]{} \vec{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/\sqrt{30} \\ -2/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix}$$

$\underline{\lambda_3 = 0:}$

$$A - 0I = \begin{bmatrix} -5 & -2 & -1 \\ -2 & -2 & 2 \\ -1 & 2 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{array}{l} x = -z \\ y = 2z \\ z \text{ free} \end{array} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 2z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\xrightarrow[\text{Normalize}]{} \vec{u}_3 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Thus $A = QDQ^T$ where

$$D = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{30} & -1/\sqrt{6} \\ 1/\sqrt{5} & -2/\sqrt{30} & 2/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}$$

21 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ■7 *Administrative Use Only*

Find a QR-decomposition of the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

$\vec{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

First perform Gram-Schmidt to the columns of A:

$\vec{v}_1 = x_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ $\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 1 \\ 2/3 \\ 2/3 \\ 4/3 \end{bmatrix}$ $\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \begin{bmatrix} -8/11 \\ 2/11 \\ 2/11 \\ 4/11 \end{bmatrix}$

Normalize:

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$ $\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} 3/\sqrt{33} \\ 2/\sqrt{33} \\ 2/\sqrt{33} \\ 4/\sqrt{33} \end{bmatrix}$ $\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \begin{bmatrix} -4/\sqrt{22} \\ 1/\sqrt{22} \\ 1/\sqrt{22} \\ 2/\sqrt{22} \end{bmatrix}$

$$\Rightarrow Q = \begin{bmatrix} 0 & 3/\sqrt{33} & -4/\sqrt{22} \\ 1/\sqrt{3} & 2/\sqrt{33} & 1/\sqrt{22} \\ 1/\sqrt{3} & 2/\sqrt{33} & 1/\sqrt{22} \\ -1/\sqrt{3} & 4/\sqrt{33} & 2/\sqrt{22} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 0 & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 3/\sqrt{33} & 2/\sqrt{33} & 2/\sqrt{33} & 4/\sqrt{33} \\ -4/\sqrt{22} & 1/\sqrt{22} & 1/\sqrt{22} & 2/\sqrt{22} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 1/\sqrt{3} & \sqrt{3} \\ 0 & 11/\sqrt{33} & 3/\sqrt{33} \\ 0 & 0 & 4/\sqrt{22} \end{bmatrix}$$

$\Rightarrow A = QR$ for Q and R as above.

 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ■7 *Administrative Use Only*

Find a singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find eigenvalues of $A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$:

$$\det(A^TA - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left((2-\lambda)^2 - 1\right) - 1\left((2-\lambda)+1\right) + 1\left(-1-(2-\lambda)\right)$$

$$= -\lambda^3 + 6\lambda^2 - 9\lambda = -\lambda(\lambda-3)^2 \qquad \Rightarrow \quad \lambda_1 = 3 \quad \lambda_2 = 3 \quad \lambda_3 = 0$$

$$\Rightarrow \quad \sigma_1 = \sqrt{3} \qquad \sigma_2 = \sqrt{3} \qquad \sigma_3 = 0$$

Eigenvalues: $\underline{\lambda_1 = \lambda_2 = 3}$ $\quad A^TA - 3I = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x = y+z \\ y, z \text{ free} \end{matrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y+z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\Leftarrow$ Not orthogonal must use Gram-Schmidt

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$\Downarrow$ normalize

$$\vec{v}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$\underline{\lambda_3 = 0:}$

$$A^TA - 0I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x = -z \\ y = z \\ z \text{ free} \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \vec{v}_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

To find $\vec{u}_3$ we must find a vector orthogonal to $\vec{u}_1$ and $\vec{u}_2$:

$$\left.\begin{aligned}\vec{u}_1\cdot\vec{u}_3 &= \begin{bmatrix}1/\sqrt{6}\\1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{\sqrt{6}}x+\frac{1}{\sqrt{6}}y+\frac{2}{\sqrt{6}}z=0\\ \vec{u}_2\cdot\vec{u}_3 &= \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y=0\end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c}1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} & 0\\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\end{array}\right]$$

$$\Rightarrow \begin{aligned}x&=-z\\y&=-z\\z&\text{ free}\end{aligned} \qquad \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-z\\-z\\z\end{bmatrix}=z\begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

$$\Rightarrow \vec{u}_3 = \begin{bmatrix}-1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$$

$$\Rightarrow A = U\Sigma V^T$$

where $U=\begin{bmatrix}1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3}\\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3}\\ 2/\sqrt{6} & 0 & 1/\sqrt{3}\end{bmatrix}$ $\qquad \Sigma=\begin{bmatrix}\sqrt{3} & 0 & 0\\ 0 & \sqrt{3} & 0\\ 0 & 0 & 0\end{bmatrix}$

and $V=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}$