Math 213 - Winter 2020
Practice Final – KEY
April 2020

Time: 5 hours

Instructions (for the final):

- Complete all exam questions from this sheet, and enter your answers in WebAssign, in the exam "Final Exam".
- This exam is closed book.
- Calculators, notes, books, online resources, and help from others is not allowed.
- Please do not communicate with others about this exam until after the exam period has closed.

Part I: Multiple Choice Questions: *(4 points each) Questions marked with a ♣ may have more than one correct answer. Enter **all** correct answers into the space provided on WebAssign. The other questions have one correct answer. Choose the best answer for each multiple choice question.* ***Correct answers underlined and in blue.***

1. ♣ Let A be a 3×4 matrix corresponding to the coefficient matrix of a system with variables x_1, x_2, x_3, x_4 given by $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Which of the following must be true? Mark all that apply

   A. $\underline{rank(A) = 3}$.

   B. $\text{Null}(A) = \{\mathbf{0}\}$.

   C. **$\underline{x_2 \text{ is a free variable.}}$**

   D. **$\underline{A\boldsymbol{x} = \boldsymbol{b} \text{ is consistent for all } \boldsymbol{b} \in \mathbb{R}^3}$**

   E. The columns of A are linearly independent.

   F. $row(A) = \mathbb{R}^3$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects $\boldsymbol{x}$ onto the line $y = 2x$. What is the standard matrix $[T]$ for this linear transformation?

   A. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

   B. $\underline{\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}$

   C. $\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

   D. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

   E. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

   F. $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. Let A, B, C, D, X be $n \times n$ invertible matrices. If $AXB^{-1} + 2C = D$, solve for the matrix X.

   A. $X = A^{-1}BD - 2A^{-1}BC$

   B. $X = ADB^{-1} - 2ACB^{-1}$

   C. $X = A^{-1}DB - 2A^{-1}CB$

   D. $X = DBA^{-1} - 2CBA^{-1}$

   E. $X = BDA^{-1} - 2BCA^{-1}$

   F. $X = 2ADB - ACB$

4. ♣ Let A be an $n \times n$ invertible matrix. Which of the following must be true? Mark all that apply.

   A. 0 is an eigenvalue of A.

   B. The rows of A are linearly dependent.

   C. $Null(A) = \{0\}$.

   D. $col(A) = \mathbb{R}^n$.

   E. $rank(A) = 0$.

   F. $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent for some $\boldsymbol{b} \in \mathbb{R}^n$.

   G. **The reduced row echelon form of A is the identity**

   H. **A is a product of elementary matrices.**

5. If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -4$ what is $\det \begin{bmatrix} a & b & c \\ 2d-3a & 2e-3b & 2f-3c \\ g & h & i \end{bmatrix}$?

   A. 6

   B. -6

   C. 8

   D. -8

   E. 2

   F. -2

   G. 0

6. ♣ Let A be an $n \times n$ which is orthogonally diagonalizable. Which of the following must be true?

   A. $A^T A = I$.

   B. $A^T = A$.

   C. **There is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.**

   D. $nullity(A)$ **is the multiplicity of the eigenvalue** 0.

   E. $rank(A) = n$.

   F. A^{-1} **is also orthogonally diagonalizable.**

   G. **Eigenvectors from distinct eigenspaces of A are orthogonal**

7. Compute the characteristic polynomial of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

   A. $-\lambda^3 + 5\lambda^2 + \lambda + 2$

   B. $-\lambda^3 + 2\lambda^2 - 2\lambda$

   C. $-\lambda^3 - 3\lambda^2 + 2\lambda + 3$

   D. $-\lambda^3 + 4\lambda^2 - 3\lambda - 8$

   E. $-\lambda^3 + 2\lambda^2 + \lambda - 2$

   F. $-\lambda^3 + \lambda^2 - 3\lambda - 1$

   G. None of the above

8. The matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (the same matrix as in the problem above) is similar to which of the following diagonal matrices?

A. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

E. $\underline{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

F. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

G. This matrix is not diagonalizable.

H. The matrix is diagonalizable, but not similar to any of the above.

9. ♣ If A is an 5×5 matrix with eigenvalues $-1, -1, 1, 1, 4$ and corresponding eigenvectors $\boldsymbol{x}_1, x_2, \boldsymbol{x}_3, \boldsymbol{x}_4, x_5$, which of the following is true? Mark all that apply.

A. $\underline{A(3\boldsymbol{x}_5) = 4(3\boldsymbol{x}_5)}$ **.**

B. **$\underline{16}$ <u>is an eigenvalue of</u> $\underline{A^2}$.**

C. A is not invertible.

D. **<u>The eigenvalues of A^{-1} are $-1, -1, 1, 1, 1/4$.</u>**

E. **<u>$\{\boldsymbol{x}_1, \boldsymbol{x}_3, \boldsymbol{x}_5\}$ is a linearly independent set.</u>**

F. $A + I$ is invertible.

10. ♣ Let A be a 5×4 matrix with linearly independent columns. Which of the following statements must be true? Mark all that apply.

A. $\operatorname{rank} A = 4$

B. $\operatorname{rank} A = 5$

C. **The equation $A\mathbf{x} = \mathbf{0}$ has a unique solution.**

D. The equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

E. The equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^5$.

F. **There is a $\mathbf{b} \in \mathbb{R}^5$ such that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution.**

G. The columns of A span $\mathbb{R}^5$.

H. **The rows of A span $\mathbb{R}^4$.**

I. The rows of A are linearly independent.

11. ♣ Let U be an $n \times n$ orthogonal matrix. Which of the following statements must be true? Mark all that apply.

A. **The columns of U are all unit vectors.**

B. U has n distinct eigenvalues.

C. There is a nonzero vector $\mathbf{x}$ with $\|U\mathbf{x}\| = 0$.

D. **The rows of U span $\mathbb{R}^n$.**

E. $\operatorname{rank} U = n$

F. $\det U = 1$

G. $\dim(\operatorname{Null} U)^{\perp} = n$

H. $U^T = U$

12. Let S be a subspace of $\mathbb{R}^3$ and let $\mathbf{y}$ be a vector given by

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \end{bmatrix}.$$

Then $\mathbf{y}$ can be written as $\mathbf{y} = \text{proj}_S\,\mathbf{y} + \text{perp}_S\,\mathbf{y}$ where $\text{proj}_S\,\mathbf{y}$ is a vector in S and $\text{perp}_S\,\mathbf{y}$ is a vector in $S^{\perp}$. Find $\text{perp}_S\,\mathbf{y}$.

A. $\text{perp}_S\,\mathbf{y} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$

B. $\text{perp}_S\,\mathbf{y} = \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ -\frac{2}{5} \\ 0 \end{bmatrix}$

C. $\text{perp}_S\,\mathbf{y} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}$

D. $\text{perp}_S\,\mathbf{y} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

E. $\text{perp}_S\,\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

Part II: Short Answer Questions: *Enter your answers into the space provided.*

13. True or **False**: The set of all vectors lying on the line $y = 3x + 4$ is a subspace of $\mathbb{R}^2$.

14. True or **False**: The dimension of $span(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ must be 3.

15. **True** or False: If A is similar to B, then $3A^{-1}$ is similar to $3B^{-1}$.

16. **True** of False: The matrix $\begin{bmatrix} -1 & 2 \\ 2 & 5 \end{bmatrix}$ is orthogonally diagonalizable.

17. True or **False**: If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$, and $S = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $S^\perp = \text{span}\{\mathbf{v}_3\}$.

18. True or **False**: If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of vectors, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent.

19. **True** or False: If a vector $\mathbf{v}$ in $\mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$, then $\mathbf{v} = \mathbf{0}$.

20. **True** or False: If S is a subspace of $\mathbb{R}^n$ and $\mathbf{y}$ is a vector in $S^\perp$, then $\text{perp}_S\, \mathbf{y} = \mathbf{y}$.

21. If A and B are 4×4 matrices and $\det(A) = 9$ and $\det(B) = -4$, then $\det(-3B^2A^{-2}) =$ **16**.

22. If A is a 2×2 matrix with eigenvalues 0 and 2 and corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ respectively, then $A^6 = \frac{1}{5}\begin{bmatrix} 256 & -128 \\ -128 & 64 \end{bmatrix}$

23. Write the spectral decomposition of $\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$. $5\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} - 5\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$

24. Let A be a matrix with singular value decomposition

$$\begin{aligned} A &= U\Sigma V^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Find the outer product form of the above singular value decomposition of A. $3\begin{bmatrix} 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$

Part III: Short Answer Questions: *Enter your answers into the space provided in WebAssign.*

25. Let W be the subspace of $\mathbb{R}^4$ given by

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Find a basis for W and a basis for $W^\perp$. What are the values of $\dim W$ and $\dim W^\perp$?

Solution. *$W = row(A)$ where $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & 1 & -1 \\ -3 & -5 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.*

Row reduction gives $\begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$.

Basis for W is $\underline{\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}}$

Since $(row(A))^\perp = Null(A)$, basis for $W^\perp$ is $\underline{\left\{ \begin{bmatrix} 6 \\ -4 \\ 3 \\ 1 \end{bmatrix} \right\}}$

Dimension of W is $\underline{3}$. Dimension of $W^\perp$ is $\underline{1}$.

26. Find the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -6 & 1 & 0 \\ -18 & 3 & 1 \end{bmatrix}$$

Solution. *We row reduce* $\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ -6 & 1 & 0 & 0 & 1 & 0 \\ -18 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 4 & -3 & 3 & 1 & 0 \\ 0 & 12 & -8 & 9 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 4 & -3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & -3 & 1 \\ 0 & 4 & 0 & 3 & -8 & 3 \\ 0 & 0 & 1 & 0 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1/4 & -1 & 1/4 \\ 0 & 1 & 0 & 3/4 & -2 & 3/4 \\ 0 & 0 & 1 & 0 & -3 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/8 & -1/2 & 1/8 \\ 0 & 1 & 0 & 3/4 & -2 & 3/4 \\ 0 & 0 & 1 & 0 & -3 & 1 \end{array}\right]$

So $A^{-1} = \underline{\begin{bmatrix} 1/8 & -1/2 & 1/8 \\ 3/4 & -2 & 3/4 \\ 0 & -3 & 1 \end{bmatrix}}$

27. Let S be the subspace

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Use the Gram-Schmidt procedure to find an orthogonal basis for S.

Solution. $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

$$\boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{-1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 5/4 \\ -3/4 \\ -1/4 \end{bmatrix}$$

$$\boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{-2}{44} \begin{bmatrix} -3 \\ 5 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4/11 \\ 8/11 \\ 4/11 \\ 16/11 \end{bmatrix}$$

Orthogonal basis $\underline{\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3/4 \\ 5/4 \\ -3/4 \\ -1/4 \end{bmatrix}, \begin{bmatrix} 4/11 \\ 8/11 \\ 4/11 \\ 16/11 \end{bmatrix} \right\}}$

28. Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 5 & 0 \\ -4 & 0 & 8 \end{bmatrix}.$$

In other words, find a diagonal matrix D and an orthogonal matrix Q such that $A = QDQ^T$.

Solution. *Eigenvalues of A:* $\lambda = 10, 5, 0$.

Corresponding eigenvectors: $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$$\underline{D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \quad \underline{Q = \begin{bmatrix} -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix}}$$

29. Find a singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ -3 & -1 \\ 0 & 0 \end{bmatrix}.$$

Solution. $A^T A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$.

Eigenvalues of $A^T A$*:* $\lambda_1 = 16, \lambda_2 = 4$*, corresponding eigenvectors:* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

So $\sigma_1 = \sqrt{16} = 4$, $\sigma_2 = \sqrt{4} = 2$, $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\underline{U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}, \quad \underline{\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}, \quad \underline{V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

Then $A = U\Sigma V^T$.