Math 213 - Winter 2020
Practice Final – KEY
April 2020

Time: 5 hours

Instructions (for the final):

- Complete all exam questions from this sheet, and enter your answers in WebAssign, in the exam "Final Exam".
- This exam is closed book.
- Calculators, notes, books, online resources, and help from others is not allowed.
- Please do not communicate with others about this exam until after the exam period has closed.

Part I: Multiple Choice Questions: *(4 points each) Questions marked with a ♣ may have more than one correct answer. Enter **all** correct answers into the space provided on WebAssign. The other questions have one correct answer. Choose the best answer for each multiple choice question.*

1. ♣ Let A be a 3×4 matrix corresponding to the coefficient matrix of a system with variables x_1, x_2, x_3, x_4 given by $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Which of the following must be true? Mark all that apply

   A. $rank(A) = 3$.

   B. $\text{Null}(A) = \{\mathbf{0}\}$.

   C. $\boldsymbol{x}_2$ is a free variable.

   D. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for all $\boldsymbol{b} \in \mathbb{R}^3$

   E. The columns of A are linearly independent.

   F. $row(A) = \mathbb{R}^3$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects $\boldsymbol{x}$ onto the line $y = 2x$. What is the standard matrix $[T]$ for this linear transformation?

   A. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

   B. $\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

   C. $\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

   D. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

   E. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

   F. $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. Let A, B, C, D, X be $n \times n$ invertible matrices. If $AXB^{-1} + 2C = D$, solve for the matrix X.

A. $X = A^{-1}BD - 2A^{-1}BC$

B. $X = ADB^{-1} - 2ACB^{-1}$

C. $X = A^{-1}DB - 2A^{-1}CB$

D. $X = DBA^{-1} - 2CBA^{-1}$

E. $X = BDA^{-1} - 2BCA^{-1}$

F. $X = 2ADB - ACB$

4. ♣ Let A be an $n \times n$ invertible matrix. Which of the following must be true? Mark all that apply.

A. 0 is an eigenvalue of A.

B. The rows of A are linearly dependent.

C. $Null(A) = \{0\}$.

D. $col(A) = \mathbb{R}^n$.

E. $rank(A) = 0$.

F. $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent for some $\boldsymbol{b} \in \mathbb{R}^n$.

G. The reduced row echelon form of A is the identity

H. A is a product of elementary matrices.

5. If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -4$ what is $\det \begin{bmatrix} a & b & c \\ 2d - 3a & 2e - 3b & 2f - 3c \\ g & h & i \end{bmatrix}$?

A. 6

B. -6

C. 8

D. -8

E. 2

F. -2

G. 0

6. ♣ Let A be an $n \times n$ which is orthogonally diagonalizable. Which of the following must be true?

A. $A^T A = I$.

B. $A^T = A$.

C. There is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

D. $nullity(A)$ is the multiplicity of the eigenvalue 0.

E. $rank(A) = n$.

F. A^{-1} is also orthogonally diagonalizable.

G. Eigenvectors from distinct eigenspaces of A are orthogonal

7. Compute the characteristic polynomial of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

A. $-\lambda^3 + 5\lambda^2 + \lambda + 2$

B. $-\lambda^3 + 2\lambda^2 - 2\lambda$

C. $-\lambda^3 - 3\lambda^2 + 2\lambda + 3$

D. $-\lambda^3 + 4\lambda^2 - 3\lambda - 8$

E. $-\lambda^3 + 2\lambda^2 + \lambda - 2$

F. $-\lambda^3 + \lambda^2 - 3\lambda - 1$

G. None of the above

8. The matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (the same matrix as in the problem above) is similar to which of the following diagonal matrices?

A. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

F. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

G. This matrix is not diagonalizable.

H. The matrix is diagonalizable, but not similar to any of the above.

9. ♣ If A is an 5×5 matrix with eigenvalues $-1, -1, 1, 1, 4$ and corresponding eigenvectors $\boldsymbol{x}_1, x_2, \boldsymbol{x}_3, \boldsymbol{x}_4, x_5$, which of the following is true? Mark all that apply.

A. $A(3\boldsymbol{x}_5) = 4(3\boldsymbol{x}_5)$.

B. 16 is an eigenvalue of A^2.

C. A is not invertible.

D. The eigenvalues of A^{-1} are $-1, -1, 1, 1, 1/4$.

E. $\{\boldsymbol{x}_1, \boldsymbol{x}_3, \boldsymbol{x}_5\}$ is a linearly independent set.

F. $A + I$ is invertible.

10. ♣ Let A be a 5×4 matrix with linearly independent columns. Which of the following statements must be true? Mark all that apply.

A. $\operatorname{rank} A = 4$

B. $\operatorname{rank} A = 5$

C. The equation $A\mathbf{x} = \mathbf{0}$ has a unique solution.

D. The equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

E. The equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^5$.

F. There is a $\mathbf{b} \in \mathbb{R}^5$ such that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution.

G. The columns of A span $\mathbb{R}^5$.

H. The rows of A span $\mathbb{R}^4$.

I. The rows of A are linearly independent.

11. ♣ Let U be an $n \times n$ orthogonal matrix. Which of the following statements must be true? Mark all that apply.

A. The columns of U are all unit vectors.

B. U has n distinct eigenvalues.

C. There is a nonzero vector $\mathbf{x}$ with $\|U\mathbf{x}\| = 0$.

D. The rows of U span $\mathbb{R}^n$.

E. $\operatorname{rank} U = n$

F. $\det U = 1$

G. $\dim(\operatorname{Null} U)^{\perp} = n$

H. $U^T = U$

12. Let S be a subspace of $\mathbb{R}^3$ and let $\mathbf{y}$ be a vector given by

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \end{bmatrix}.$$

Then $\mathbf{y}$ can be written as $\mathbf{y} = \text{proj}_S\,\mathbf{y} + \text{perp}_S\,\mathbf{y}$ where $\text{proj}_S\,\mathbf{y}$ is a vector in S and $\text{perp}_S\,\mathbf{y}$ is a vector in $S^{\perp}$. Find $\text{perp}_S\,\mathbf{y}$.

A. $\text{perp}_S\,\mathbf{y} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$

B. $\text{perp}_S\,\mathbf{y} = \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ -\frac{2}{5} \\ 0 \end{bmatrix}$

C. $\text{perp}_S\,\mathbf{y} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}$

D. $\text{perp}_S\,\mathbf{y} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

E. $\text{perp}_S\,\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

Part II: Short Answer Questions: *Enter your answers into the space provided.*

13. True or False: The set of all vectors lying on the line $y = 3x + 4$ is a subspace of $\mathbb{R}^2$.

14. True or False: The dimension of $span(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ must be 3.

15. True or False: If A is similar to B, then $3A^{-1}$ is similar to $3B^{-1}$.

16. True of False: The matrix $\begin{bmatrix} -1 & 2 \\ 2 & 5 \end{bmatrix}$ is orthogonally diagonalizable.

17. True or False: If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$, and $S = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $S^\perp = \text{span}\{\mathbf{v}_3\}$.

18. True or False: If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of vectors, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent.

19. True or False: If a vector $\mathbf{v}$ in $\mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$, then $\mathbf{v} = \mathbf{0}$.

20. True or False: If S is a subspace of $\mathbb{R}^n$ and $\mathbf{y}$ is a vector in $S^\perp$, then $\text{perp}_S\,\mathbf{y} = \mathbf{y}$.

21. If A and B are 4×4 matrices and $\det(A) = 9$ and $\det(B) = -4$, then $\det(-3B^2A^{-2}) =$ ____________.

22. If A is a 2×2 matrix with eigenvalues 0 and 2 and corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ respectively, then $A^6 =$ __________________

23. Write the spectral decomposition of $\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$.

24. Let A be a matrix with singular value decomposition

$$\begin{aligned} A &= U\Sigma V^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Find the outer product form of the above singular value decomposition of A.

Part III: Short Answer Questions: *Enter your answers into the space provided in WebAssign.*

25. Let W be the subspace of $\mathbb{R}^4$ given by

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Find a basis for W and a basis for $W^\perp$. What are the values of $\dim W$ and $\dim W^\perp$?

26. Find the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -6 & 1 & 0 \\ -18 & 3 & 1 \end{bmatrix}$$

27. Let S be the subspace

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Use the Gram-Schmidt procedure to find an orthogonal basis for S.

28. Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 5 & 0 \\ -4 & 0 & 8 \end{bmatrix}.$$

In other words, find a diagonal matrix D and an orthogonal matrix Q such that $A = QDQ^T$.

29. Find a singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ -3 & -1 \\ 0 & 0 \end{bmatrix}.$$