Math 213 - Winter 2020
Final Exam – KEY
April 2020

Time: 5 hours

Instructions:

- Complete all exam questions from this sheet, and enter your answers in WebAssign, in the exam "Final Exam".
- This exam is closed book.
- Calculators, notes, books, online resources, and help from others is not allowed.
- Please do not communicate with others about this exam until after the exam period has closed.

Part I: Multiple Choice Questions: *(4 points each) Questions marked with a ♣ may have more than one correct answer. Enter **all** correct answers into the space provided in Learning Suite. The other questions have one correct answer. Choose the best answer for each multiple choice question.*

1. ♣ Let A be a 3×4 matrix corresponding to the coefficient matrix of a system with variables x_1, x_2, x_3, x_4 whose reduced row echelon form (RREF) is $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Which of the following must be true? Mark all that apply

   A. $rank(A) = 2$.

   B. **A has a non-trivial null space.**

   C. Variables x_3 and x_4 are free.

   D. **If $A\boldsymbol{x} = \boldsymbol{b}$ is consistent, it has infinitely many solutions.**

   E. The columns of A span $\mathbb{R}^3$.

   F. $dim(row(A)) = 4$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects $\boldsymbol{x}$ over the line $y = -x$. What is the standard matrix $[T]$ for this linear transformation?

   A. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

   B. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   C. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

   D. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

   E. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   F. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

3. Let A, B, C, X be $n \times n$ matrices. If $(2AX^{-1}B^{-1})^{-1} = C$, solve for the matrix X.

A. $X = \frac{1}{2}B^{-1}CA$

B. $X = 2B^{-1}AC$

C. $X = \frac{1}{2}B^{-1}AC$

D. $X = 2B^{-1}CA$

E. $X = 2A^{-1}CB$

F. $X = \frac{1}{2}ACB^{-1}$

G. $X = \frac{1}{2}ACB$

H. $X = 2AB^{-1}C$

4. ♣ Let A be an $n \times n$ invertible matrix. Which of the following must be true? Mark all that apply.

A. $\det(A) = 0$.

B. **The columns of A are linearly independent.**

C. $nullity(A) = 0$**.**

D. $row(A) = \mathbb{R}^n$**.**

E. $rank(A) < n$.

F. $A\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions for all $\boldsymbol{b} \in \mathbb{R}^n$.

G. The reduced row echelon form of A has a row of all 0.

H. **A^T is also invertible.**

5. If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6$ what is $\det \begin{bmatrix} b & 2a & c \\ e & 2d & f \\ h & 2g & i \end{bmatrix}$?

A. 12

B. -12

C. 6

D. -6

E. 2

F. -2

G. 0

6. ♣ Let A be a *symmetric* 4×4 matrix with eigenvalues $0, 1, 1, 3$ and corresponding eigenvectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$. Which of the following is true? Mark all that apply.

   A. $A^T = A^{-1}$.

   B. $A^T = A$.

   C. $rank(A) = 3$.

   D. $\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 = 0$

   E. A is invertible.

   F. **A is orthogonally diagonalizable.**

   G. **Eignvalue $\lambda = 1$ has geometric multiplicity 2.**

   H. $\boldsymbol{x}_1 - \boldsymbol{x}_2 = 0$.

7. Compute the characteristic polynomial of the matrix $\begin{bmatrix} -5 & 0 & 4 \\ 0 & 3 & 0 \\ -8 & 0 & 7 \end{bmatrix}$.

   A. $-\lambda^3 + 5\lambda^2 - 3\lambda - 9$

   B. $-\lambda^3 + 5\lambda^2 - 6\lambda + 2$

   C. $-\lambda^3 + 5\lambda^2 - 6\lambda$

   D. $-\lambda^3 + 5\lambda^2 - 7\lambda + 3$

   E. $-\lambda^3 + 5\lambda^2 - 6\lambda - 8$

   F. $-\lambda^3 + 5\lambda^2 + 6\lambda$

   G. None of the above

8. The matrix $\begin{bmatrix} -5 & 0 & 4 \\ 0 & 3 & 0 \\ -8 & 0 & 7 \end{bmatrix}$ (the same matrix as in the problem above) is similar to which of the following diagonal matrices?

A. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 6 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

F. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

G. This matrix is not diagonalizable.

H. The matrix is diagonalizable, but not similar to any of the above.

9. ♣ If A is an $n \times n$ matrix with eigenvalue $\lambda = 3$ and corresponding eigenvector $\boldsymbol{x}$, which of the following is true? Mark all that apply.

A. $2\boldsymbol{x}$ is and eigenvector of A with eigenvalue 6.

B. **9 is and eigenvalue of A^2.**

C. **If A is invertible, $A^{-1}\boldsymbol{x} = \frac{1}{3}\boldsymbol{x}$.**

D. **$(A - 3I)\boldsymbol{x} = 0$.**

E. $\boldsymbol{x} = 0$.

F. $rank(A - 3I) = n$.

10. ♣ Let A be a 3×5 matrix with rank 3. Which of the following statements must be true? Mark all that apply.

A. The equation $A\mathbf{x} = \mathbf{0}$ has a unique solution.

B. **The equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.**

C. **The equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^3$.**

D. There is a $\mathbf{b} \in \mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution.

E. **The columns of A span $\mathbb{R}^3$.**

F. The columns of A are linearly independent.

G. The rows of A span $\mathbb{R}^5$.

H. **The rows of A are linearly independent.**

11. ♣ Let U be an $n \times n$ orthogonal matrix. Which of the following statements must be true? Mark all that apply.

A. **U has orthonormal rows.**

B. **The columns of U are a basis for $\mathbb{R}^n$.**

C. U is orthogonally diagonalizable.

D. **$UU^T = I_n$, where I_n is the $n \times n$ identity matrix.**

E. $\operatorname{Nullity} U = n$

F. **If λ is an eigenvalue of U, then $|\lambda| = 1$.**

G. **$\mathbf{x} \cdot \mathbf{y} = U\mathbf{x} \cdot U\mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.**

H. $(\operatorname{Row} U)^{\perp} = \mathbb{R}^n$

12. Let S be a subspace of $\mathbb{R}^3$ and let $\mathbf{y}$ be a vector given by

$$S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathbf{y} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}.$$

Find the orthogonal projection $\operatorname{proj}_S \mathbf{y}$ of the vector $\mathbf{y}$ onto S.

A. $\operatorname{proj}_S \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

B. $\operatorname{proj}_S \mathbf{y} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

C. $\operatorname{proj}_S \mathbf{y} = \begin{bmatrix} \frac{5}{2} \\ \frac{7}{2} \\ 0 \end{bmatrix}$

D. **$\operatorname{proj}_S \mathbf{y} = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \\ -2 \end{bmatrix}$**

E. $\operatorname{proj}_S \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$

Part II: Short Answer Questions: *Enter your answers into the space provided.*

13. **True** or False: The set of all vectors lying on the line $y = -2x$ is a subspace of $\mathbb{R}^2$.

14. True or **False**: If A is an $n \times n$ matrix with $\det(A) = 0$, the columns of A are a basis for $\mathbb{R}^n$.

15. **True** or False: If A is similar to B, then A^T is similar to B^T.

16. True of **False**: The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is orthogonally diagonalizable.

17. True or **False**: Let $S = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$ be a subspace of $\mathbb{R}^4$. If $\mathbf{v} \cdot \mathbf{w}_1 = 0$ and $\mathbf{v} \cdot \mathbf{w}_2 = 0$, then $S^\perp = \text{span}\{\mathbf{v}\}$.

18. **True** or False: The set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

is an orthogonal set.

19. **True** or False: Let S be a subspace of $\mathbb{R}^n$, and suppose that $\mathbf{v}$ is in both S and $S^\perp$. Then $\mathbf{v} = \mathbf{0}$.

20. **True** or False: If S is a subspace of $\mathbb{R}^n$ and $\mathbf{y}$ is a vector in S, then $\text{proj}_S\, \mathbf{y} = \mathbf{y}$.

21. If A and B are 3×3 matrices and $\det(A) = 3$ and $\det(B) = 5$, then $\det(-2B^{-1}A^T) =$ **-24/5**.

22. If A is a 2×2 matrix with eigenvalue 1 and corresponding eigenvector $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and eignvalue -1 with eigenvector $\boldsymbol{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, then $A^{101}(4\boldsymbol{x} + 2\boldsymbol{y}) = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

23. Find the singular values of the matrix

$$\begin{bmatrix} -16 & 0 \\ 0 & 4 \end{bmatrix}.$$

$\sigma_1 = 16, \;\; \sigma_2 = 4$

Part III: Short Answer Questions: *Enter your answers into the space provided in WebAssign.*

24. Let W be the subspace of $\mathbb{R}^4$ given by

$$W = \text{span}\left\{ \begin{bmatrix}1\\2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\-2\end{bmatrix}, \begin{bmatrix}0\\2\\0\\2\end{bmatrix} \right\}.$$

Find a basis for W and a basis for $W^\perp$. What are the values of $\dim W$ and $\dim W^\perp$?

Solution. $W = row\left(\begin{bmatrix}1&2&1&0\\1&1&1&-1\\1&0&1&-2\\0&2&0&2\end{bmatrix}\right)$. *RREF is* $\begin{bmatrix}1&0&1&-2\\0&1&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$. *Basis for* W: $\underline{\left\{\begin{bmatrix}1\\2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}\right\}}$. $dim(W) = \underline{2}$

$W^\perp$ *is the null space of this matrix. Basis for* $W^\perp$: $\underline{\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\1\end{bmatrix}\right\}}$. $dim(W) = \underline{2}$

25. Find the inverse A^{-1} of the matrix

$$A = \begin{bmatrix}2&1&-3\\1&0&-2\\-2&-1&4\end{bmatrix}.$$

Solution. $\left[\begin{array}{rrr|rrr}2&1&-3&1&0&0\\1&0&-2&0&1&0\\-2&-1&4&0&0&1\end{array}\right] \sim \left[\begin{array}{rrr|rrr}1&0&-2&0&1&0\\0&1&1&1&-2&0\\0&-1&0&0&2&1\end{array}\right] \sim \left[\begin{array}{rrr|rrr}1&0&0&2&1&2\\0&1&0&0&-2&-1\\0&0&1&1&0&1\end{array}\right]$

$A^{-1} = \underline{\begin{bmatrix}2&1&2\\0&-2&-1\\1&0&1\end{bmatrix}}$

26. Let S be the subspace

$$S = \text{span}\left\{ \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix} \right\}.$$

Use the Gram-Schmidt procedure to find an orthogonal basis for S.

Solution. $\boldsymbol{u}_1 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$

$\boldsymbol{u}_2 = \begin{bmatrix}0\\1\\-1\\1\end{bmatrix} - \frac{0}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}$

$\boldsymbol{u}_3 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}0\\1\\-1\\1\end{bmatrix} = \begin{bmatrix}1/3\\1/3\\0\\-1/3\end{bmatrix}$

Orthogonal basis $\underline{\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}1/3\\1/3\\0\\-1/3\end{bmatrix}\right\}}$

27. Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}.$$

In other words, find a diagonal matrix D and an orthogonal matrix Q such that $A = QDQ^T$.

Solution. *Characteristic polynomial:* $\lambda^2 - 15\lambda + 50$.

Eigenvalues: $\lambda = 10, 5$. *Eigenvectors:* $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; *normalized:* $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$Q = \underline{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}.$$

$$D = \underline{\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}}.$$

28. Find a singular value decomposition of the matrix

$$A = \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} \\ 0 & 0 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}.$$

Solution. $A^TA = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$

Characteristic polynomial: $\lambda^2 - 20\lambda + 64$. *Eigenvalues:* $\lambda = 16, 4$. *Singular values:* $\sigma_1 = 4$, $\sigma_2 = 2$.

Eigenvectors of A^TA: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. *Normalized:* $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\boldsymbol{u}_1 = \frac{1}{\sigma_1}A\boldsymbol{v}_1 = \frac{1}{4}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \boldsymbol{u}_2 = \frac{1}{\sigma_2}A\boldsymbol{v}_2 = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$U = \underline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}},\ \Sigma = \underline{\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}},\ V = \underline{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}.$$