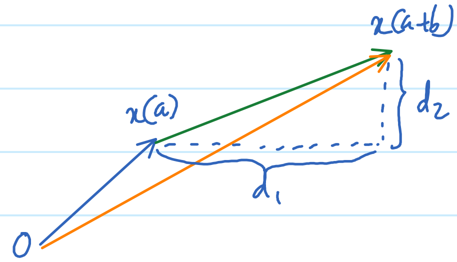
Lecture 2

Thursday, September 3, 2020

Suppose we want to keep track of our position on the plane.



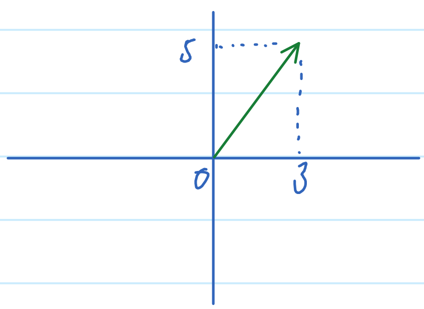
Our position is $x = x(t)$, a function of one variable (time). Note that $x(t)$ has two components: $x(t) = (x_1(t), x_2(t))$ where $x_1(t)$ is the first coordinate, $x_2(t)$ the second coordinate.

Our initial position is $x(0) = (0, 0)$. After time $t = a$, we are at position $x(a)$. After another time b, we travel d_1 (unit length) to the left and d_2 (unit length) to the right. We are then at position

$$x(a+b) = (x_1(a) + d_1,\ x_2(a) + d_2)$$

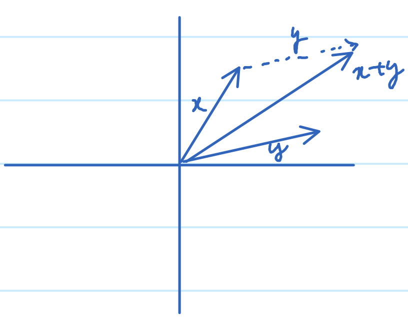
We see that in order to keep track of position, we have to keep track of both coordinates. Vector is a useful tool to represent position, where two (or more) coordinates can be tracked at once!

A pair of numbers is a vector. For example, $x = [3, 5]$ is a vector. It is represented by an arrow in a coordinate system



Note that vector x is represented as an arrow starting at the origin.

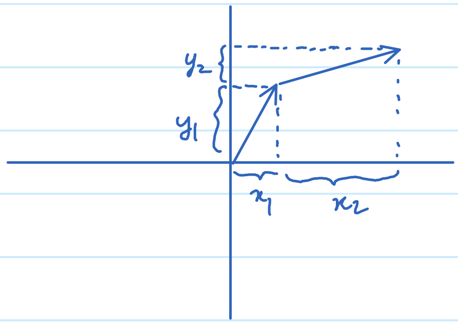
How to "add" two vectors? The example above shows how to add two vectors.



To add vector x and y, we move y so that the starting point of y coincides the ending point of x. Then the new ending point of y is the ending point of $x+y$.

In coordinates, if $x = [x_1, x_2]$ and $y = [y_1, y_2]$ then

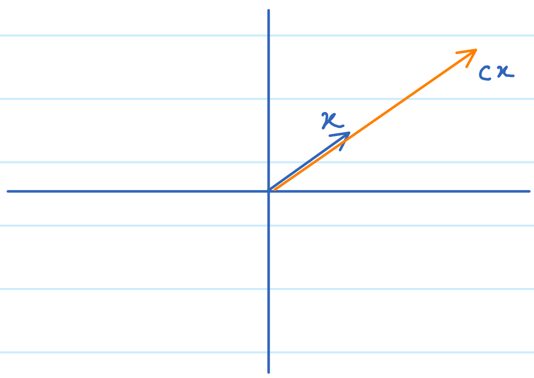
$$x + y = [x_1 + y_1, x_2 + y_2]$$



We can also scale a vector:

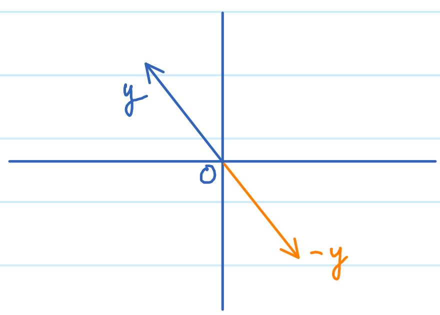
$c \in \mathbb{R}$

$cx = [cx_1, cx_2]$



Rule of thumb: add or scale vectors by adding or scaling component by component.

Subtraction: $x - y = x + (-y)$



One can generalize these rules to "general" vectors. In general, a vector can have more than two components. A vector in $\mathbb{R}^n$ has n components:

$$x = [x_1, x_2, \ldots, x_n].$$

Addition:

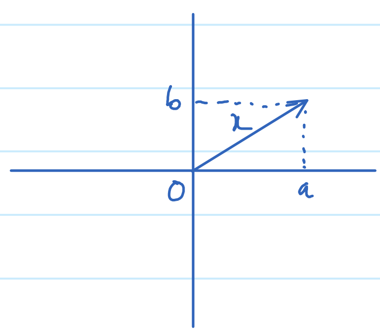
$$x + y = [x_1, \ldots, x_n] + [y_1, \ldots, y_n] = [x_1 + y_1, \ldots, x_n + y_n]$$

Scaling: $cx = c[x_1, x_2, \ldots, x_n] = [cx_1, cx_2, \ldots, cx_n]$

If motions can only happen in 3-dimensional space, why do we care about vectors with more than 3 components?

$\rightsquigarrow$ The notion of vector makes computations simpler, even if a vector doesn't represent a real motion in space. Some difficult problems can be made simpler by a procedure called "linearization", in which all calculations can be done on vectors. We will see further examples later.

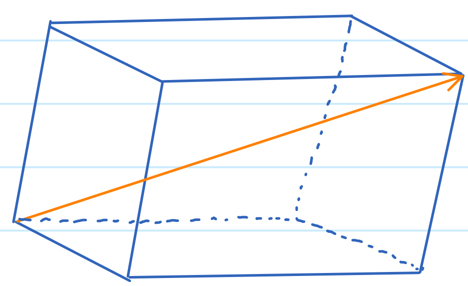
Length of a vector



If $x = [a, b]$ then the length / magnitude / modulus of x is denoted as

$$\|x\| = \sqrt{a^2 + b^2}$$

Food for thought: how do we compute the distance between two opposite corners of a box?



Remark:

Points on the plane is usually denoted by uppercase letters and parentheses. For example, $A = (2, 5)$. Vectors are often denoted by lowercase letters and square brackets. For example, $a = [2, 5]$. The addition of points doesn't make as much sense as addition of vectors. That is why we only define addition of vectors.