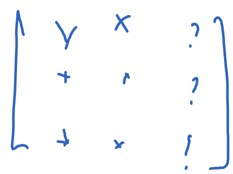
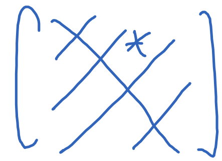
Lecture 28

Thursday, December 10, 2020 4:00 PM

A ... symmetric matrix

$A =$ 

$A =$ 

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

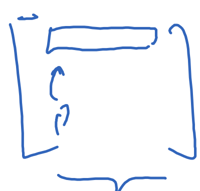
$\pm i$

$A = A^T$

- real eigenvalues
- diagonalizable $\quad A = PDP^{-1}$
- P can be chosen to be orthogonal.

Spectral decomposition $\rightarrow$ special case of S.V.D.

$$A = \underbrace{\square}_{\text{rank } 1} + \underbrace{\square}_{\text{rank } 1} + \dots + \underbrace{\square}_{\text{rank } 1}$$



rank 1

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix}$$

$$\mathrm{col}(A) = \{ \underbrace{c_1 v_1 + c_2 v_2 + \dots + c_n v_n}_{A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}} : c_1, \dots, c_n \in \mathbb{R} \}$$

$$\rightarrow \mathrm{col}(A) = \{ Av : v \in \mathbb{R}^n \}$$

If $\mathrm{rank}(A) = 1$ then

$$Av = cw$$

$$A = PDP^{-1} = PDP^T = \begin{array}{c} \begin{array}{ccc} a_{11} & a_{12} & a_{1n} \end{array} \\ \begin{array}{c} a_{21} \\ \\ a_{n1} \end{array} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \end{array} \begin{bmatrix} \lambda_1 & & \\ 0 & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} — & v_1^T & — \\ — & v_2^T & — \\ & \vdots & \\ — & v_n^T & — \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} — & v_1^T & — \\ — & v_2^T & — \\ & \vdots & \\ — & v_n^T & — \end{bmatrix}$$

$$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \dots + \lambda_n v_n v_n^T$$ ← Spectral dcm

number

$$\underbrace{v_1}_{n\times 1} \underbrace{v_1^T}_{1\times n} \underbrace{a}_{n\times 1} = \text{number} \times v_1$$

Ex:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longleftarrow$$

$0, 0, 3$

$$E(0) = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{u_2} \right\}$$

$$A = \cancel{\lambda_1 v_1 v_1^T} + \cancel{\lambda_2 v_2 v_2^T} + \lambda_3 v_3 v_3^T$$

$$A = 3\underbrace{\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}}$$

$$E(3) = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{u_3} \right\}$$

$$A = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & 3 \end{bmatrix} \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix}^{-1}$$

$$\overset{v_1}{u_1}, \overset{v_2}{u_2}, u_3$$

$$v_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$v_2 = \frac{1}{\ell}(u_2 - \text{proj}_{u_1} u_2) = \frac{1}{\ell}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$$

$$= \frac{1}{\ell} \underbrace{\begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}} = \sqrt{\frac{2}{3}} \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$v_3 = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

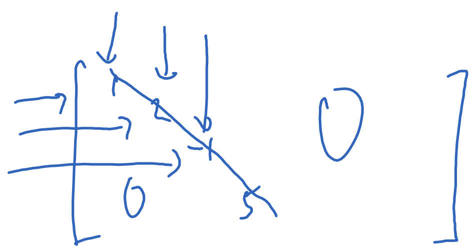
A symmetric $\longrightarrow$ spectral decom.

$$A = P D P^T$$

A $m \times n$ singular value decomposition.

$$\underset{\substack{\text{any} \\ m \times n}}{A} = \underbrace{U}_{\substack{m \times m \\ \text{orthogonal}}} \underbrace{\Sigma}_{\substack{m \times n \\ \text{diagonal matrix}}} \underbrace{V^T}_{\substack{n \times n \\ \text{orthor}}}$$

How to get U, Σ, V?

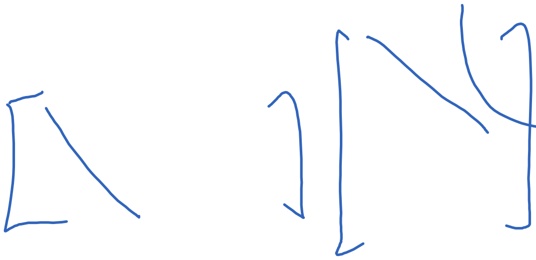


$$A = U \Sigma V^T$$

$$A^T = V \Sigma^T U^T$$

$$A^T A = V \Sigma^T \underbrace{U^T U}_{I} \Sigma V^T = V \Sigma^T \Sigma V^T$$

$$= V \underbrace{\Sigma^T \Sigma}_{\substack{n \times n \\ D}} V^{-1}$$



$$A^T A = V D V^{-1}$$

A^TA $\longrightarrow$ orthogonally diag

$$A^TA = V D V^T$$

$$\Sigma^T \Sigma = D$$

$$D = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{\sigma_1} & & & \\ & \sqrt{\sigma_2} & & 0 \\ 0 & & \ddots & \\ & & & \sqrt{\sigma_n} \end{bmatrix}$$

$A = U \Sigma V^T$ How to get U?

$$\underline{AV} = U \Sigma \underbrace{V^T V}_{I} = U\Sigma = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sqrt{\sigma_1} & & & \\ & \sqrt{\sigma_2} & & 0 \\ 0 & & \sqrt{\sigma_n} & \\ & & & 0 \cdots 0 \end{bmatrix}$$

$$= \begin{bmatrix} | & | & \\ \sqrt{\sigma_1} u_1 & \sqrt{\sigma_2} u_2 & \cdots \\ | & | & \end{bmatrix}$$

$\sqrt{\sigma_1}\, u_1$ = first col. of AV $\longrightarrow u_1$

$u_2, \ldots$