Final Exam: Some problems for review

The Final exam is comprehensive, with emphasis on the material covered after the second midterm. You should review all assigned homework problems in those sections (6.6 - 7.7). In addition, you should review the midterm exams. Their solution keys are on the course website. The Final exam is a closed-book exam. No calculators are allowed. If your calculations look tedious, it is likely that you are going on a wrong track, or that you make a computational mistake earlier. The following power series will be provided on the exam:

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!}, \\ \ln(1+x) &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \\ \sin x &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!}, \\ \cos x &= \sum_{k=0}^{\infty} (-1)^{k} \frac{x^{2k}}{(2k)!}, \\ \frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k. \end{aligned}$$

The table of Laplace transform on page 252 of the textbook will also be provided. The following problems are some additional problems to help you review.

1. Which of the following sets of vectors (of functions) are linearly DEPENDENT?

   a. $x^{(1)} = \begin{bmatrix} 1 \\ t+1 \end{bmatrix}$, $x^{(2)} = \begin{bmatrix} 2 \\ 2(t+1) \end{bmatrix}$

   b. $x^{(1)} = \begin{bmatrix} 1 \\ t+1 \end{bmatrix}$, $x^{(2)} = \begin{bmatrix} t \\ t(t+1) \end{bmatrix}$

   c. $x^{(1)} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$, $x^{(2)} = \begin{bmatrix} 0 \\ \sin t \end{bmatrix}$

   d. $x^{(1)} = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$, $x^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $x^{(3)} = \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}$

2. What is the domain of a power series $\sum a_n x^n$?

   a. The set of all x's such that the power series converges.

   b. The set of all x's such that the power series absolutely converges.

3. The function $x(t) = (2t-1)e^{2t} + 2e^t$ is a solution to which of the following ODEs?

   a. $x''' + 5x'' - 8x' - 4x = 0$

   b. $x''' - 5x'' + 8x' - 4x = 0$

   c. $x''' - 5x'' + 8x' - 4x = e^{2t}$

   d. $x''' + 5x'' + 8x' + 4x = e^{2t}$

4. In the phase portrait of the ODE $x' = \begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix} x$, the origin is an equilibrium of what type?

a. Node

b. Saddle point

c. Spiral point

d. Center point

5. If A is a 2×2 matrix with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$ and with corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. What is A^4 ?

   a. $\begin{bmatrix} -18 & -34 \\ 15 & 31 \end{bmatrix}$

   b. $\begin{bmatrix} -18 & -30 \\ 15 & 31 \end{bmatrix}$

   c. $\begin{bmatrix} 14 & 30 \\ -15 & -31 \end{bmatrix}$

   d. $\begin{bmatrix} -14 & -30 \\ 15 & 31 \end{bmatrix}$

6. A mass of 2 lb is hanged on a spring with spring coefficient 8 lb/in. Suppose the damping coefficient is 8 lb $\cdot$ s/in. This motion is

   a. Overdamped

   b. Underdamped

   c. Critically damped

   d. Undamped

7. What is the domain of the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{(n+1)!}$$

   a. $(-\infty, \infty)$

   b. $[1, 3]$

   c. $(1, 5)$

   d. $[-1, 5)$

8. Let x be the solution to the ODE $x' = e^x(\cos x + 1)$ satisfying $x(2) = 6$. What is the limit of $x(t)$ as $t \to \infty$?

   a. ∞

   b. $-\infty$

   c. π

   d. 3π

9. Which of the vector fields in Figure 1 is the direction field of the ODE $x' = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} x$?

10. Which of the vector fields in Figure 2 is the direction field of the ODE $x' = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} x$?

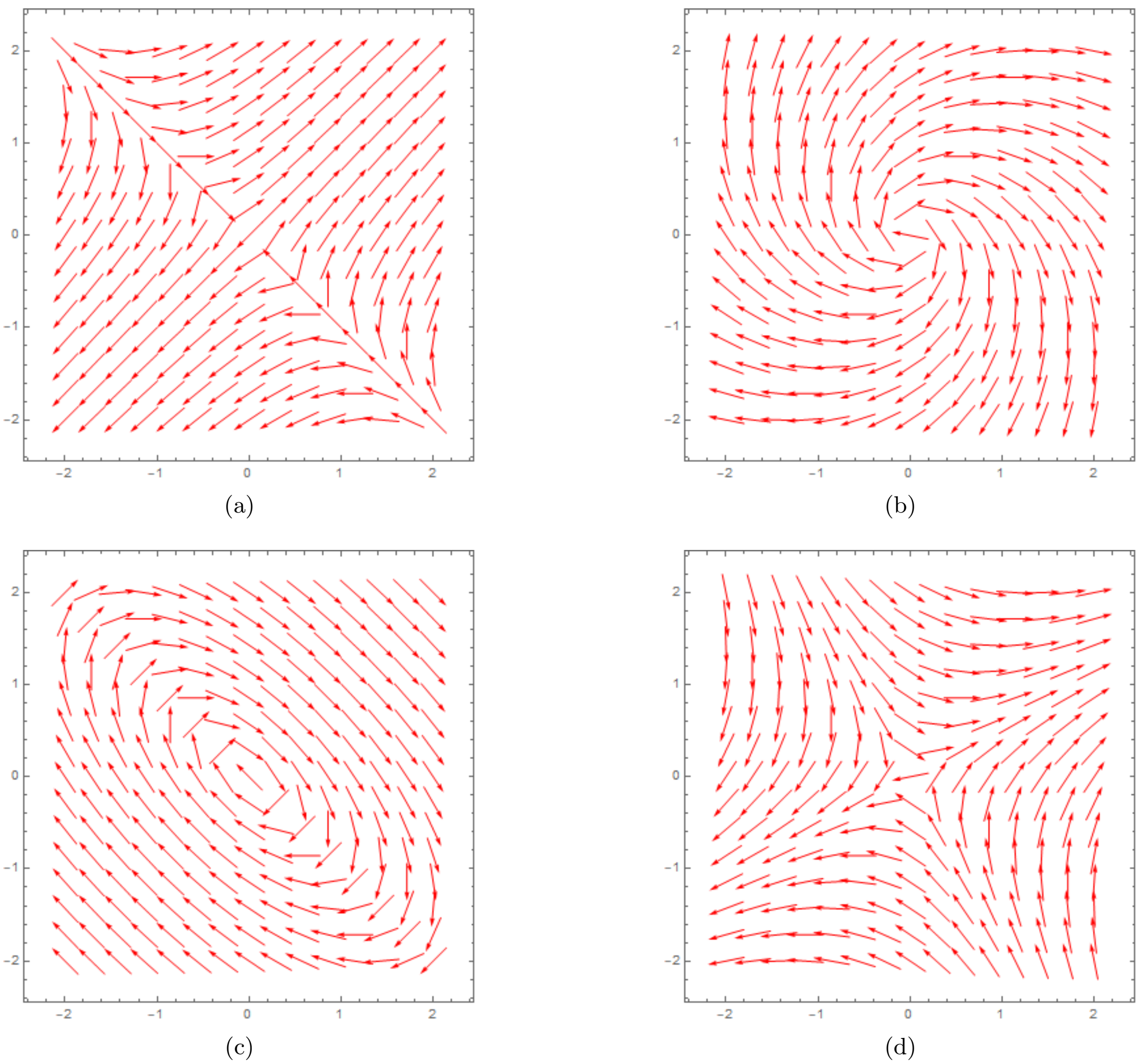
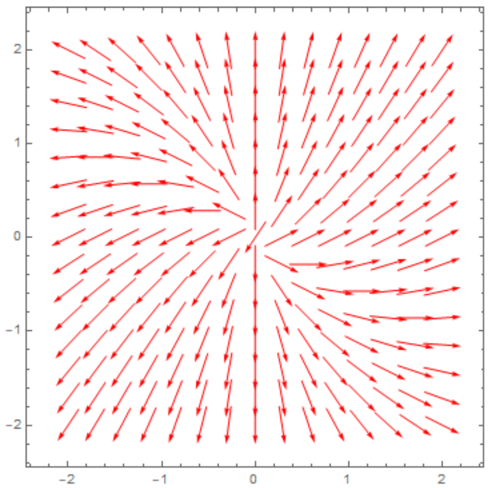
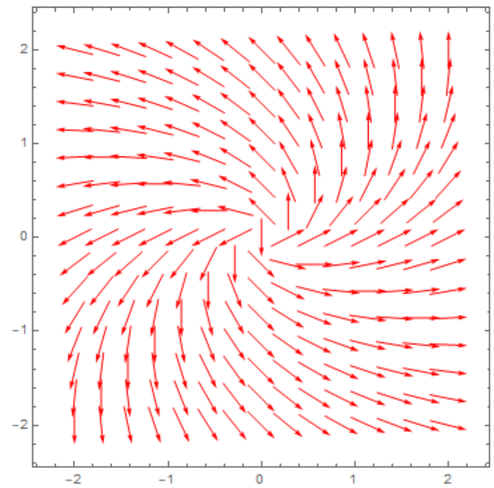


Figure 1: Direction fields (Problem 9)



(a)



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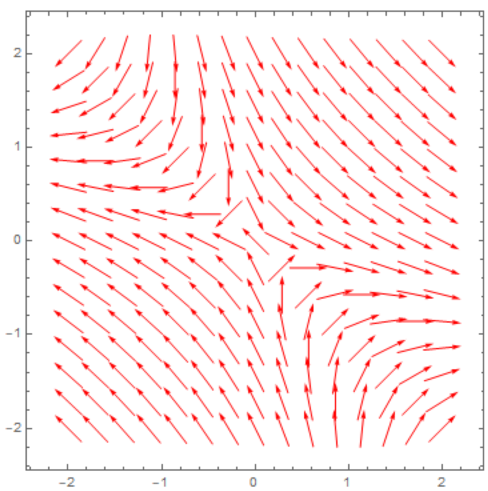

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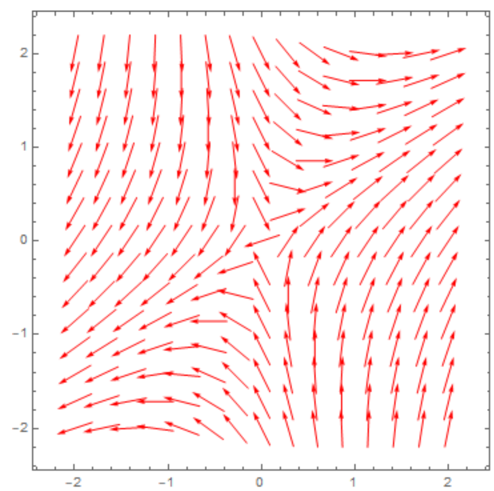

(d)

Figure 2: Direction fields (Problem 10)

11. A mass of 2 lb is hanged on a spring with spring coefficient 8 lb/in. The mass is acted on by a periodic external force $F = 3\cos(\omega t)$. Suppose the motion is undamped. Find the value of ω which causes resonance. Then find the response of the system given the initial conditions $u(0) = u'(0) = 0$.

12. Let $x^{(1)} = \begin{bmatrix} t \\ e^t \end{bmatrix}$ and $x^{(2)} = \begin{bmatrix} t^2 \\ \cos t \end{bmatrix}$. Find the Wronskian of $x^{(1)}$ and $x^{(2)}$. Can they both solve an ODE of the form $x' = P(t)x$?

13. Let $v(t) = \begin{bmatrix} t+i \\ 2+it \end{bmatrix}$ and $a(t) = e^{2t+3i}$. Write $a(t)v(t)$ in complex standard form $v_1(t) + iv_2(t)$.

14. Is the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ diagonalizable? If write A in the form PDP^{-1} where P is an invertible matrix and D is a diagonal matrix.

15. Find the Laplace transform of $f(t) = (t-1)e^{-t}$.

16. Find the inverse Laplace transform of $F(s) = \frac{1}{(s+1)^2(s^2+1)}$.
Hint: use convolution or partial fractional decomposition.

17. Convert the following system into a single ODE.

$$\begin{cases} x_1' &= 2x_1 + x_2 + t \\ x_2' &= x_1 - 2x_2 + t^2 \end{cases}$$

18. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Find e^{At}. Then solve the equation $x' = Ax$ with the initial condition

$$x(0) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

19. Let $A(t) = \begin{bmatrix} t^2 & t \\ t-1 & t+1 \end{bmatrix}$. Determine $A'(1)$ and $\int_1^2 A(t)^2 dt$.

20. Solve the initial value problem

$$y'' + y = \begin{cases} 1 & \text{if } \ 0 \le t < 2, \\ t-1 & \text{if } \ t \ge 2 \end{cases}$$

with initial conditions $y(0) = y'(0) = 0$. Hint: use either variation of parameters or Laplace transform.

21. Solve the initial value problem

$$y''' - 2y'' - y' - 2y = t$$

with initial conditions $y(0) = 0$, $y'(0) = 1$, $y''(0) = 1$.

Answer keys:

1. a
2. a
3. b
4. c
5. d
6. c
7. a
8. d
9. a
10. b
11. $\omega = 3$ and $u(t) = \frac{3}{8} t \sin(2t)$
12. $t \cos t - t^2 e^t$. No
13. $v_1(t) = \begin{bmatrix} e^{2t} t\cos(3) - e^{2t}\sin(3) \\ 2e^{2t}\cos(3) - e^{2t} t \sin(3) \end{bmatrix}$ and $v_2(t) = \begin{bmatrix} e^{2t}\cos(3) + e^{2t} t \sin(3) \\ e^{2t} t\cos(3) + 2e^{2t}\sin(3) \end{bmatrix}$
14. Yes.
$$A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$
15. $-\frac{s}{(s+1)^2}$
16. $\frac{1}{2} e^{-t} \left(t - e^t \cos(t) + 1\right)$
17. $x_1'' - 4x_1' + 3x_1' - t^2 + 2t = 0$. One can also write an equation in terms of x_2 only. One equation is sufficient.