Lecture 2

Thursday, September 3, 2020

What information can we extract from a differential equation (without solving it)?

Let us consider an example: $y' = y(2-y)$.

It is a short form for $y'(t) = y(t)(2-y(t))$.

One can check that the function $y_1(t) = t$ doesn't solve the equation. After trying other functions, one can see that it is hard to get a solution for this problem. We will learn how to solve it later (the Separation of Variable method). But let us examine what information we can infer about the solution $y = y(t)$ without knowing an exact formula of y.

$$y' = y(2-y) \qquad (1)$$

We first see that if we know y at any given time, we will know the slope of y at that time. For example, if $y(2) = 5$ then

$$y'(2) = y(2)(2-y(2)) = 5(2-5) = -15.$$

In more practical language, y is a quantity depending on time, Equation (1) says that the rate of change of y is determined by the value of y at the current time.

$\rightsquigarrow$ Whenever a practical problem involves the rate of change of some quantity in time, one usually ends up having an ODE.

Let us add an initial condition: $y(0) = 1$

$$\begin{cases} y' = y(2-y) \\ y(0) = 1 \end{cases}$$

With some simple calculations, one can find all derivatives of y at 0, i.e. $y'(0), y''(0), y'''(0), \ldots$. Indeed, by differentiating both sides of (1) we get

$$\begin{aligned} y'' &= 2y' - 2yy' \qquad \text{(chain rule)} \\ &= 2y(2-y) - 2y^2(2-y) \end{aligned}$$

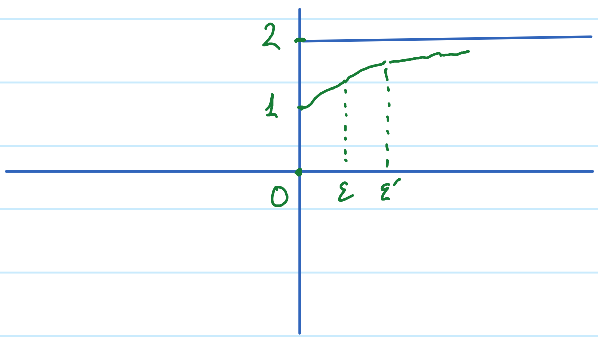
Thus,

$$y''(0) = 2y(0)(2-y(0)) - 2y(0)^2(2-y(0)) = \ldots$$

Another observation: we know that

$$y'(0) = y(0)(2-y(0)) = 1(2-1) = 1 > 0$$

Thus, y is increasing near $t=0$.



After some small interval of time, say ε, the solution y increases from 1 to some values less than 2. Now that $1 < y(\varepsilon) < 2$, we have.

$$y'(\varepsilon) = y(\varepsilon)(2-y(\varepsilon)) > 0.$$

This implies that y is still increasing at $t=\varepsilon$. We then repeat this process. The solution will never cross the line $y=2$. Otherwise, it will be equal to 2 for all time (which is a contradiction because we start with $y(0)=1$). Because y is increasing and is always bounded from above by 2, it has a horizontal asymptote.

$$\lim_{t\to\infty} y(t) = a$$

The slope of y seems to be close to 0 as $t\to\infty$.

$$\underbrace{y'(t)}_{\downarrow\, 0} = \underbrace{y(t)}_{\downarrow\, a}\,(2-\underbrace{y(t)}_{\downarrow\, a}) \quad \text{as } t\to\infty$$

Thus, $0 = a(2-a)$. Because $a>1$, $a=2$.

We see that y must be an increasing function and have the horizontal line $y=2$ as an asymptote. This is known even before we can solve for y analytically.

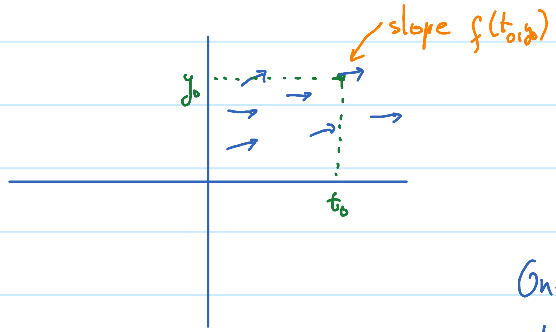
Let us consider a visual way to solve ODEs of the form

$$y' = f(t,y)$$

where f is a given function of two variables.

We want to solve for $y=y(t)$, which is a curve.

On the ty-diagram, if the curve passes through a point, say (t_0, y_0), then we know the slope of the curve at that point:



$$y'(t_0) = f(t_0, y(t_0)) = f(t_0, y_0).$$

One can draw a map of arrows: at each location (t_0, y_0) on the diagram, we place a short arrow of slope $f(t_0, y_0)$.

A solution curve must be tangent to all the arrows it passes through.

This map is called a direction field.

Ex:

$$f(t,y) = t+y$$

$$y' = f(t,y) = t+y$$

In Mathematica, try the following commands:

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VectorPlot[{1, t+y}, {t,-2,2}, {y,-2,2}]
StreamPlot[{1, t+y}, {t,-2,2}, {y,-2,2}]
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