Lecture 25

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$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + g_1(t)$$

$$x_2' = p_{21}(t)x_1 + p_{22}x_2 + \dots + p_{2n}x_n + g_2(t)$$

$$\vdots$$

$$x_n' = p_{n1}x_1 + p_{n2}x_2 + \dots + p_{nn}x_n + g_n(t)$$

$$x' = \underbrace{P(t)}_{\begin{bmatrix} p_{11} & p_{12} & \dots \\ & & \\ & & \end{bmatrix}_{n\times n}} \underbrace{x}_{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} + \underbrace{g}_{\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}} \leftarrow$$

If P is a diagonal matrix then there is no coupling

$$P = \begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{x'} = \underbrace{\begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix}}_{P(t)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$$x_1' = t x_1 + \sin t$$

$$x_2' = t^2 x_2 + \cos t$$

P ... diagonal matrix $\to$ done!

P is not diagonal : if P is diagonalizable then

$$P = Q \underset{\substack{\uparrow \\ \text{diagonal matrix}}}{D} Q^{-1}$$

$$x' = Px + g = QD\underbrace{Q^{-1}x}_{y} + g$$

$$\underbrace{Q^{-1}x'}_{?\ y'} = Q^{-1}(QDy + g) = Dy + Q^{-1}g$$

$$y = Q^{-1}x \quad \cdots\cdots \quad y' = Q^{-1}x'$$

$$(fg)' = f'g + fg'$$

$$P = \begin{bmatrix} t & 2t \\ t^2 & -t \end{bmatrix} = QDQ^{-1}$$

$$y' = Dy + Q^{-1}g$$

$x' = P x + \cancel{g}$

$x' = \underset{\sim}{P} x$ constant matrix

$\cancel{\dfrac{x'}{x} = P}$

$x' = ax$

$\dfrac{dx}{dt} = ax \longrightarrow \dfrac{dx}{x} = a\,dt$

$\ln|x| = at + C$

$|x| = e^{at+C}$

$x = \pm e^{C} e^{at}$

$x = k e^{at}$

2×1

$x = v e^{at} = e^{at} v$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x' = Px$

$P = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

What are a and v so that $x = e^{at} v$ solve the eq. $x' = Px$

$\longrightarrow x' = \left(e^{at} v\right)' = a e^{at} v$

$Px = P e^{at} v = e^{at} P v$

$\underline{av = Pv}$

Suppose v is a constant vector :

$$P = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

If v is an eigenvector & a is the corresponding eigenvalue then

$$x = e^{at} v \text{ is a sol. of } x' = Px$$

$$P - \lambda I_2 = \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$$

$$|P - \lambda I_2| = (1-\lambda)^2 - 4 = (1-\lambda)^2 - 2^2 = (-1-\lambda)(3-\lambda)$$

$$\lambda_1 = -1, \quad \lambda_2 = 3$$

$\lambda_1 = -1$

$$(P - \lambda_1 I_2)v = 0 \Longleftrightarrow Pv = \lambda_1 v$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v = \begin{bmatrix} a \\ b \end{bmatrix} \qquad a + \frac{1}{2} b = 0$$

$$b = -2, \quad a = 1$$

$$v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \lambda_1 = -1$$

$$x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \lambda_2 = 3$$

$$x^{(1)} = e^{\lambda_1 t} v_1 = e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$x^{(2)} = e^{\lambda_2 t} v_2 = e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Thm Consider the equation $x' = P(t)x$, where P is an $n\times n$ matrix. There are n linearly independent solutions $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$. Moreover, any solution to the ODE is a linear comb. of $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$.

$$\underbrace{x^{(1)}, x^{(2)}, \ldots}_{\text{lin. ind.}} \qquad x^{(1)} = \begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{bmatrix}$$

if the eq. $c_1 x^{(1)} + c_2 x^{(2)} + \ldots + c_n x^{(n)} = 0 \quad \forall t$

const.

implies $c_1 = c_2 = \cdots = c_n = 0$.

$$x^{(1)} = \begin{bmatrix} 1 \\ t \end{bmatrix}, \qquad x^{(2)} = \begin{bmatrix} t \\ t^2 \end{bmatrix} = t\begin{bmatrix} 1 \\ t \end{bmatrix} = t x^{(1)}$$

$$t x^{(1)} - x^{(2)} = 0$$

How can we determine if given vectors are lin. ind.

Ex.
$$v_1 = \begin{bmatrix} t \\ 2t \end{bmatrix}, \quad v_2 = \begin{bmatrix} t^2 \\ 3t \end{bmatrix}$$

$$\det\left(\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}\right) = \det\left(\begin{bmatrix} t & t^2 \\ 2t & 3t \end{bmatrix}\right) = 3t^2 - 2t^3$$

$c_1 v_1 + c_2 v_2 = 0$ for $(c_1, c_2) \neq (0,0)$ $\rightsquigarrow$ $\det\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = 0 \quad \forall t$

pick $t = 1$. $\det \neq 0 \rightsquigarrow v_1, v_2$ are lin. ind.

$$v_1 = \begin{bmatrix} 1 \\ t \end{bmatrix}, \quad v_2 = \begin{bmatrix} t \\ t^2 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1 & t \\ t & t^2 \end{bmatrix}\right) = 0 \quad \forall t$$

$$f^{(1)} = \begin{bmatrix} f_{11} \\ f_{12} \\ \vdots \\ f_{1n} \end{bmatrix}, \quad f^{(2)} = \begin{bmatrix} \\ \\ \end{bmatrix}, \ \ldots \ f^{(n)} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$W[f^{(1)}, \ldots, f^{(n)}] = \det\left(\underbrace{\begin{bmatrix} | & | & & | \\ f^{(1)} & f^{(2)} & \cdots & f^{(n)} \\ | & | & & | \end{bmatrix}}\right) \neq 0 \text{ for some } t$$