Lecture 25

Thursday, December 3, 2020 12:19 PM

$$y'' + 3y' + y = 0$$

$y_1 \quad y_2$

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W[f, g] = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

If $W \neq 0$ at some t then f, g are lin. ind

Suppose f, g are linearly dep.

$\rightarrow c_1 f(t) + c_2 g(t) = 0 \quad \forall t$

$c_1 f'(t) + c_2 g'(t) = 0 \quad \forall t$

$$\rightarrow \underbrace{\begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix}}_{\substack{A \\ \text{non-invertible}}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{A^{-1} A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = A^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

* If f & g are lin. dep. then $\underbrace{\begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}}_{W[f,g]} = 0 \quad \forall t.$

If A then B.

If not B then not A

* If $W[f,g] \neq 0$ for some t then f and g are lin. ind.

$$y'' + 3y' + y = 0$$

y_1, y_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$z_1 = y$

$z_2 = y'$

$z_1' = y' = z_2$

$z_2' = y'' = -3y' - y = -3z_2 - z_1$

$$\underbrace{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}'}_{z'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{z}$$

$$z' = Pz$$

$z^{(1)} = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix}$

$z^{(2)} = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}$

$$W = \begin{vmatrix} | & | \\ z^{(1)} & z^{(2)} \\ | & | \end{vmatrix}$$

$y''' + y'' + y' + y = 0 \rightsquigarrow y_1, y_2, y_3$

$$W[y_1, y_2, y_3] = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$z_1 = y$

$z_2 = y'$

$z_3 = y''$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}' = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$z' = P z$$

$$\underline{z}^{(1)} = \begin{bmatrix} y_1 \\ y_1' \\ y_1'' \end{bmatrix}$$

$$W = \begin{vmatrix} | & | & | \\ z^{(1)} & z^{(2)} & z^{(3)} \\ | & | & | \end{vmatrix}$$

$$W = \begin{vmatrix} | & | & & | \\ x^{(1)} & x^{(2)} & \cdots\cdots & x^{(n)} \\ | & | & & | \end{vmatrix}$$

$x' = P x$

$\underline{x}^{(1)}, \underline{x}^{(2)}, \dots, \underline{x}^{(n)}$

If $x^{(1)}, \dots, x^{(n)}$ are lin. ind., then $\begin{bmatrix} | & & | \\ x^{(1)} & \cdots & x^{(n)} \\ | & & | \end{bmatrix}$ is called a fundamental matrix of the eq $x' = \underline{P} x$.

Coef. matrix

Thm: $x' = P(t)x$ has n lin. ind. sol $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$.

Any other sol. is a lin. comb of these sols.

$$x = c_1 x^{(1)} + c_2 x^{(2)} + \cdots + c_n x^{(n)}$$

Constant

If P is a constant matrix, with eigenvector v, and eigenvalue λ then $x(t) = e^{\lambda t} v$ is a sol. to $x' = Px$.

$$Pv = \lambda v$$

Ex: $P = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ $\quad x' = Px$

$\lambda_1 = -1$ $\quad \lambda_2 = 4$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$x^{(1)} = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad x^{(2)} = e^{4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

lin. ind

a general sol is $x = c_1 x^{(1)} + c_2 x^{(2)}$

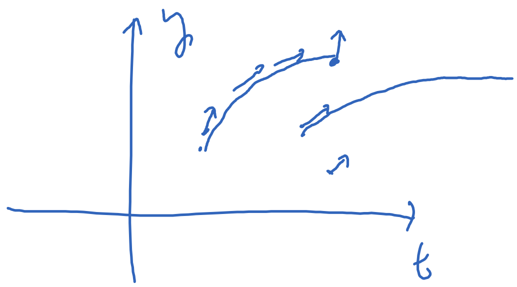
$$x = c_1 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix} c_1e^{-t}+2c_2e^{4t}\\ -c_1e^{-t}+3c_2e^{4t}\end{bmatrix}$$

$$\begin{cases} x_1 = c_1 e^{-t} + 2c_2 e^{4t} \\ x_2 = -c_1 e^{-t} + 3c_2 e^{4t} \end{cases}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t}+2e^{4t} \\ -e^{-t}+3e^{4t} \end{bmatrix}$$

ODE $\underline{y' = f(t,y)}$ $\qquad y = y(t)$

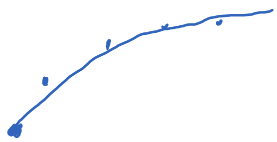
$$y' = t+y$$

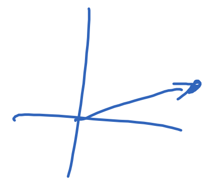


direction field

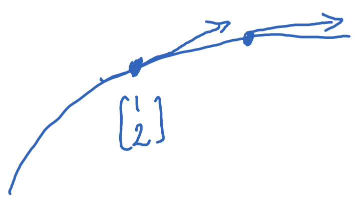
$$x' = P x \qquad P = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$





$$\underbrace{x' = P \overset{\text{point}}{x}}_{\text{vector}}$$



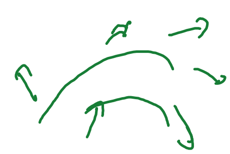
$$x = x(t)$$

$$x'(t)$$

$$x' = P\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

phase portrait



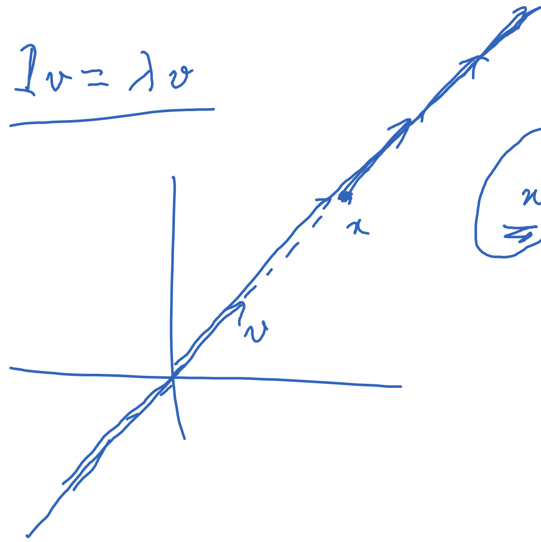
direction field

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$Px = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 2x_2 \end{bmatrix}$$

$$\sqrt{(x_1+2x_2)^2 + (3x_1+2x_2)^2}$$

$Pv = \lambda v$



$$x' = Px = P(cv)$$
$$= cPv$$
$$= c\lambda v$$