Midterm II: Some problems for review

The second midterm exam will cover Sections 3.5-8, 4.1-4, 5.1-3, 6.1-4. You should review all assigned homework problems in those sections. The exam is a closed-book exam. No calculators are allowed. If your calculations look tedious, it is likely that you are going on a wrong track, or that you make a computational mistake earlier. The following power series will be provided on the exam:

$$\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!}, \\
\ln(1+x) &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \\
\sin x &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!}, \\
\cos x &= \sum_{k=0}^{\infty} (-1)^{k} \frac{x^{2k}}{(2k)!}, \\
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k.
\end{aligned}$$

The table of Laplace transform on page 252 of the textbook will also be provided. The following problems are for review beside the homework problems.

In Problem 1 through 8, circle all correct answers. In the exam, you will not be asked to give explanations. But you should know the reason for your answers in this practice. In Problem 9 through 20, write out your answer carefully. Try not to skip any steps.

1. The general solution to the differential equation $y'' - 2y' - 3y = \cos t$ is of the form

$$y = c_1 e^{-t} + c_2 e^{3t} + A\cos t + B\sin t.$$

   a. True
   b. False

2. There is a solution to the differential equation $y'' - 4y' + 4y = e^{2t} + \cos t$ is of the form

   a. $y = Ae^{2t} + B\cos t + C\sin t$
   b. $y = Ate^{2t} + B\cos t + C\sin t$
   c. $y = At^2e^{2t} + B\cos t + C\sin t$
   d. $y = At^3e^{2t} + B\cos t + C\sin t$

3. A linear homogeneous ODE of order $n \geq 1$ with constant coefficients always has n linearly independent solutions.

   a. True
   b. False

4. A mass of 2 lb is hanged on a spring with spring coefficient 60 lb/in. Suppose the damping coefficient is 1 lb $\cdot$ s/in. Denote by $u(t)$ the displacement of the mass from its equilibrium position, measured positive in the downward direction, at time t. Then u satisfies the equation

a. $2u'' + u' + 60u = 0$

b. $2u'' - u' + 60u = 0$

c. $2u'' + u' - 60u = 0$

d. $2u'' - u' + 60u = 2g$

5. The motion described in Problem 3 is

a. Overdamped

b. Underdamped

c. Critically damped

d. Undamped

6. A mass of 2 lb is hanged on a spring with spring coefficient 18 lb/in. The mass is acted on by a periodic external force $F = 2\sin(\omega t + \pi/6)$. Suppose the motion is undamped. Then resonance happens when ω is equal to

a. 0

b. 1

c. 2

d. 3

7. The differential equation $y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y = g(t)$, where $p_1, \ldots, p_n, g$ are given functions with $g \neq 0$, is called a ______, ______ ODE.

a. linear, nonhomogeneous

b. linear, homogeneous

c. nonlinear, nonhomogeneous

d. nonlinear, homogeneous

8. The Wronskian of functions 1, $\sin t$, t^2 in that order is equal to

a. 0

b. $2\cos t + 2\sin t$

c. $2\cos t + 2t\sin t$

d. $2\cos t - 2t\sin t$

9. Solve the initial value problem

$$y'' + 3y' + 2y = \cos t, \quad y(0) = y'(0) = 1.$$

10. Solve the initial value problem

$$y'' - 2y' + y = 2e^{t/2}, \quad y(0) = y'(0) = 0.$$

11. Find the general solution of the ODE

$$y''' - 2y'' - y' + 2y = e^{4t} + \cos t.$$

12. Find the general solution of the ODE

$$y'' - 3y' + 2y = \sin(e^{-t}).$$

Hint: use variation of parameters.

13. Suppose that a linear, homogeneous ODE with constant coefficients has the characteristic polynomial $(r^2+1)^2(r+1)$. What is the ODE? Determine the general solution.

14. Find the solution of the initial value problem

$$y'' - 3y' + 2y = g(t), \quad y(0) = 0, \ y'(0) = 1,$$

where

$$g(t) = \begin{cases} 1 & \text{if} \quad 0 \le t < 2, \\ 0 & \text{if} \qquad\quad t > 2. \end{cases}$$

Hint: use either variation of parameters or Laplace transform.

15. Express the function $f(x) = \frac{5}{3+2x^2}$ as a power series centered at 0. Determine the radius of convergence.

16. Determine the domain of the function

$$f(x) = \sum_{k=1}^{\infty} \frac{(x-1)^k}{2^k k^2}.$$

17. Find the first three nonzero terms of a power series solution to the initial value problem

$$y' = 3y(1+xy), \quad y(0) = 1.$$

18. Determine whether the following integral converges or diverges.

(a)

$$\int_0^\infty \frac{e^{-t^2+1}}{t} dt$$

(b)

$$\int_1^\infty \frac{\sin t + \cos t}{t^2} dt$$

(c)

$$\int_0^1 e^{-1/t} dt$$

(d)

$$\int_1^\infty \frac{t^2-5}{t^4+3} \sin(t) dt$$

19. Give an example of a function that is analytic at 5 and has radius of convergence $\sqrt{2}$.

20. Consider the differential equation

$$(x^2+1)^2 y''' + \ln(x+2) y'' + (\sin x) y = x.$$

This equation has a power series solution centered at 0. Find a lower bound (as large as possible) of the radius of convergence.

21. Using summation notation, write the power series of function $\ln(2x^2+3)$ centered at 0. Find the radius of convergence.

Answer keys:

1. a
2. c
3. a
4. a
5. b
6. d
7. a
8. c
9. $\frac{1}{10}e^{-2t}(-16+25e^t+e^{2t}\cos t+3e^{2t}\sin t)$
10. $4e^{t/2}\left(e^{t/2}t-2e^{t/2}+2\right)$
11. $c_1e^{-t}+c_2e^t+c_3e^{2t}+\frac{1}{30}\left(e^{4t}-3\sin t+6\cos t\right)$
12. $c_1e^t+c_2e^{2t}-e^{2t}\cos\left(e^{-t}\right)$
13. $c_1e^{-t}+c_2\cos t+c_3\sin t+c_4t\cos t+c_5t\sin t$
14. $$y(t)=\begin{cases} \frac{1}{2}\left(-4e^t+3e^{2t}+1\right) & \text{if} \quad 0<t<2 \\ \frac{1}{2}\mathrm{e}^{-4+t}\left(2\mathrm{e}^2-4\mathrm{e}^4-\mathrm{e}^t+3\mathrm{e}^{4+t}\right) & \text{if} \quad t>2 \end{cases}$$
15. $$\sum_{k=0}^{\infty}\frac{5}{3}(-1)^k\left(\frac{2}{3}\right)^k x^{2k}$$
    Radius of convergence is $\sqrt{3/2}$.
16. $[-1,3]$
17. $y(x)=1+3x+6x^2+\dots$
18. (a) diverges by comparing with $1/t$. (b) converges by comparing with $2/t^2$. (c) converges by using the change of variable $u=1/t$. (d) converges by comparing with $1/t^2$.
19. $f(x)=\sum_{k=0}^{\infty}\frac{(x-5)^k}{\sqrt{2}^k}=\frac{1}{1-\frac{x-5}{\sqrt{2}}}$.
20. $R\geq 1$
21. $\ln(2x^2+3)=\ln 3+\ln\left(\frac{2x^2}{3}+1\right)=\ln 3+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\left(\frac{2x^2}{3}\right)^k}{k}=\dots$ (simplify)
    Radius of convergence is $\sqrt{3/2}$.