MATH 334, MIDTERM EXAM I, FALL 2020

Name	BYU ID

- This is a closed-book exam. Calculators are not allowed.
- For problems other than True/False or multiple choice, you must show valid arguments with all necessary steps. Mysterious answers will receive little or no credit.

Problem	Possible points	Earned points
1	5	
2-10	18	
11	10	
12	10	
13	10	
14	10	
15	10	
16	10	
Total	83	

Problem 1. (5 points) Identify the direction field corresponding to each given differential equation.

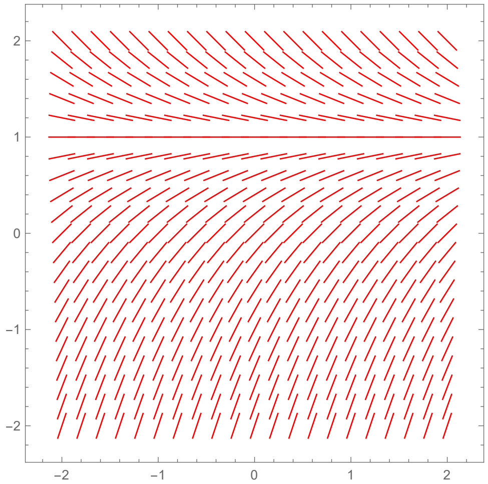
D $y' = t^2$

C $y' = ty$

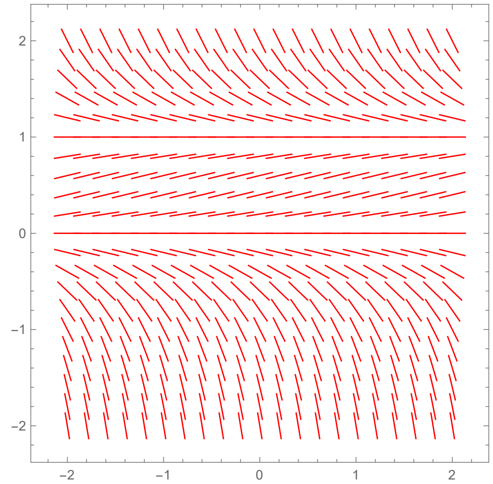
B $y' = y(1-y)$

E $y' = y(1-t)$

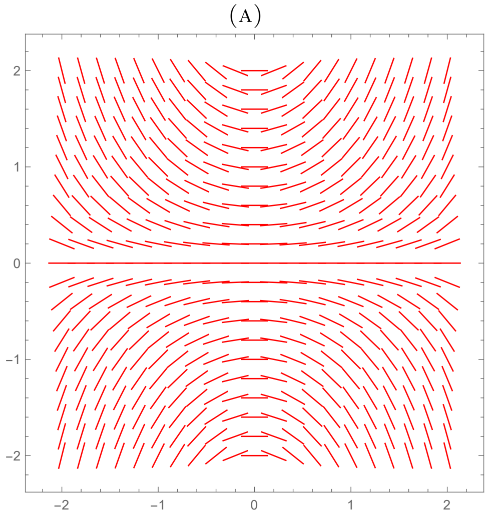
A $y' = 1 - y$



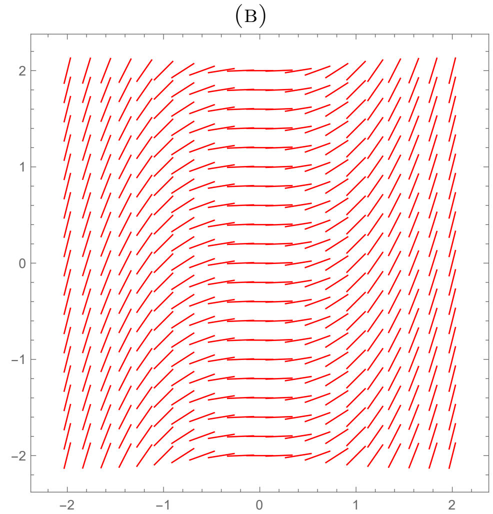
(A)



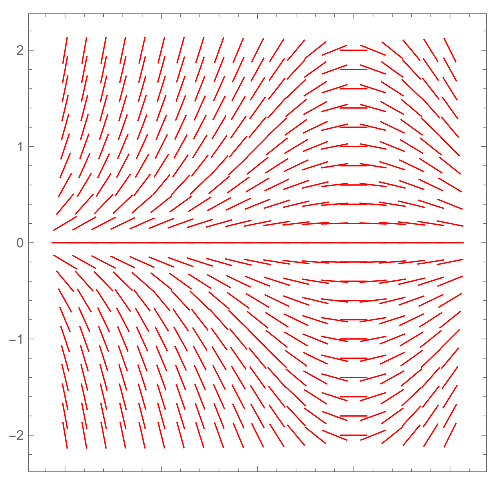
(B)



(C)

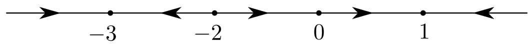


(D)



(E)

Problem 2. (2 points) An autonomous ODE $y' = f(y)$ has the following phase line. Suppose $y(2) = -1$. What is $\lim_{t\to\infty} y(t)$?



a. -3
b. -2
c. 0
d. 1
e. 2
f. ∞

Problem 3. (2 points) The ODE $y' + yy'' = 0$ is second order.

True False

Problem 4. (2 points) Every solution to the ODE $y' = \cos(y)$ is bounded. (That is, there is some number $M > 0$ such that $|y(t)| \le M$ for all t.)

True False

Problem 5. (2 points) If a function $f(t, y)$ and its partial derivative $\frac{\partial f}{\partial y}$ are continuous in the rectangle $(-1, 1) \times (0, 2)$ then the initial value problem $y' = f(t, y)$, $y(0) = 1$ has a solution on $(-1, 1)$.

True False

Problem 6. (2 points) For any functions u and v, if $W[u, v](t) = 0$ for all $t \in \mathbb{R}$ then u and v are linearly dependent.

True False

Problem 7. (2 points) The function $y = te^{2t}$ is a solution of $y'' - y' - 2y = 0$.

True False

Problem 8. (2 points) The interval where the solution to the initial value problem

$$(1 - t^2)y' + ty = \ln(t), \quad y(2) = 0$$

is certain to exist is

a. $(-\infty, 0)$
b. $(0, 1)$
c. $(1, \infty)$
d. $(0, \infty)$

Problem 9. (2 points) Let $u(t) = t^2$ and $v(t) = e^t$. Then $W[u, v](1)$ is equal to

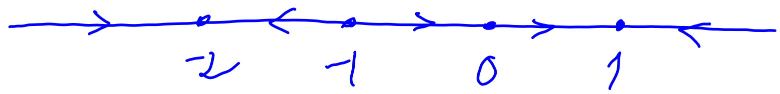
a. 0
b. 1
c. $e^2 - e$
d. $-e$

Problem 10. (2 points) The functions $y_1 = e^{2t}\cos(t)$ and $y_2 = e^{2t}\sin(t)$ are solutions of the ODE

a. $y'' + 4y' + 5y = 0$
b. $y'' - 4y' + 5y = 0$
c. $y'' + 2y' + y = 0$
d. $y'' + 2y' + 5y = 0$

Problem 11. (10 points) Consider the autonomous ODE $y' = y^2(1 - y^2)(y + 2)$. Determine all the equilibrium solutions. Draw the phase line. Determine if each equilibrium is stable, unstable, or semistable. Sketch a few approximate solutions in the ty-plane.

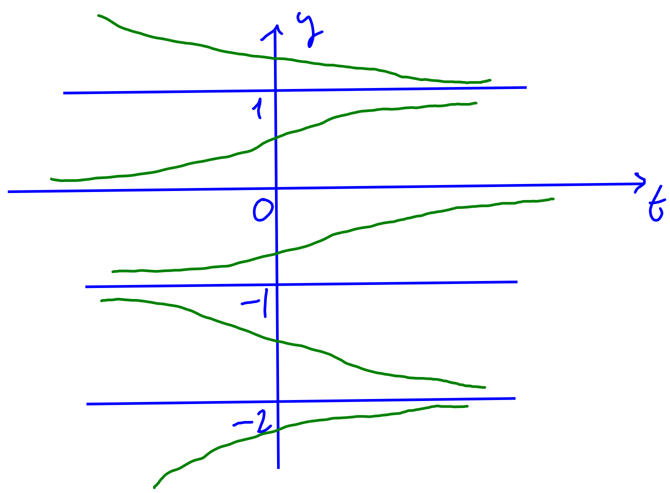
Equilibria: $0, -1, 1, -2$



Stable: $-2, 1$

unstable: -1

Semistable: 0



Problem 12. (10 points) Solve the initial value problem

$$y' = \frac{2x}{y} + \frac{1}{y}, \quad y(0) = -2.$$

$$y' = \frac{2x+1}{y}$$

$$\rightsquigarrow \quad y y' = 2x+1$$

Integrating both sides: $\frac{y^2}{2} = x^2 + x + C$

Thus, $y = \pm\sqrt{2(x^2+x+C)}$

For $x=0$, we get $-2 = \pm\sqrt{2C}$.

Thus, the minus sign is chosen and $C=2$.

$$y = -\sqrt{2(x^2+x+2)}$$

Problem 13. (10 points) Solve the initial value problem

$$x(y' + 2) = y, \quad y(1) = 2.$$

$$y' + 2 = \frac{1}{x} y$$

$$y' - \frac{1}{x} y = -2 \qquad (*)$$

Integrating factor:
$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}.$$

Multiply both sides of $(*)$ by $\frac{1}{x}$:

$$\frac{1}{x} y' - \frac{1}{x^2} y = -\frac{2}{x}$$

Then $$\left(\frac{1}{x} y\right)' = -\frac{2}{x}$$

Integrating both sides.

$$\frac{1}{x} y = -2 \ln x + C$$

$$\rightsquigarrow \quad y = -2x\ln x + Cx$$

Plug $x=1$, we get

$$2 = -2 \cdot 1 \underbrace{\ln 1}_{0} + C \cdot 1$$

Thus gives $C = 2$.

$$y = -2x\ln x + 2x$$

Problem 14. (10 points) Solve the initial value problem

$$y' = y(y-3), \quad y(0) = 2.$$

$$\frac{y'}{y(y-3)} = 1 \rightsquigarrow \int \frac{dy}{y(y-3)} = x + C$$

Decompose $\frac{1}{y(y-3)}$ as follows (fractional decomposition):

$$\frac{1}{y(y-3)} = \frac{A}{y} + \frac{B}{y-3}$$

We get $A = -\frac{1}{3}$, $B = \frac{1}{3}$.

$$\frac{1}{3}\int\left(\frac{1}{y-3} - \frac{1}{y}\right)dy = x + C$$

$$\rightsquigarrow \frac{1}{3}\left(\ln|y-3| - \ln|y|\right) = x + C$$

$$\rightsquigarrow \frac{1}{3}\ln\left|\frac{y-3}{y}\right| = x + C$$

$$\rightsquigarrow \ln\left|\frac{y-3}{y}\right| = 3x + 3C \rightsquigarrow \frac{y-3}{y} = \pm e^{3x+3C}$$

$$\rightsquigarrow \frac{y-3}{y} = k e^{3x}$$

Substituting $x = 0$, we get $\frac{2-3}{2} = k$. Thus, $k = -1/2$

$$y - 3 = k y e^{3x} \longrightarrow y(1 - k e^{3x}) = 3$$

$$\rightsquigarrow y = \frac{3}{1 + \frac{1}{2}e^{3x}}$$

Problem 15. (10 points) Solve the initial value problem

$$y'' + 4y' + 3y = 0, \quad y(0) = 1, \ y'(0) = -1.$$

Determine the limit of $y(t)$ as $t \to \infty$.

Characteristic eq: $r^2 + 4r + 3 = 0$

Two roots are $r = -1$ and $r = -3$.

The general sol. is $y = c_1 e^{-t} + c_2 e^{-3t}$.

To compute c_1, c_2, we differentiate y:

$$y' = -c_1 e^{-t} - 3c_2 e^{-3t}.$$

Because $y(0) = 1$ and $y'(0) = -1$, we have

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 - 3c_2 = -1 \end{cases}$$

From here we get $c_1 = 1$ and $c_2 = 0$.

$$y = e^{-t}$$

$$\lim_{t \to \infty} y(t) = 0$$

Problem 16. (10 points) A container initially has 500 gal of mixture of salt and water with salt concentration 0.1 lbs per gal. People purify the container by pumping fresh water into the container at a volume rate of 3 gal/min, and at the same time pumping out from the container at the rate of 2 gal/min. Assume that the mixture is well-stirred at all time. How much time will elapse before the concentration of the salt in the container is equal to 1/8 ~~reaches a half~~ of the original concentration?

$y = y(t)$ ..... amount of salt at time t.

$V = V(t)$ ... volume of the mixture at time t.

$c = c(t) = \dfrac{y(t)}{V(t)}$ ... concentration of salt at time t.

We have
$$\frac{dV}{dt} = 3 - 2 = 1$$

Thus, $V(t) = t + C$. With $V(0) = 500$, we get $C = 500$.

Then $V(t) = t + 500$.

We have
$$\frac{dy}{dt} = 0 \times 3 - \frac{y}{V} \times 2 = -\frac{2y}{t+500}$$

Thus, $\dfrac{dy}{y} = -\dfrac{2}{t+500}\,dt$

Integrate both sides: $\ln y = -2\ln(t+500) + C$

Exponentiate both sides: $y = \dfrac{k}{(t+500)^2}$ $(*)$

Because $c(0) = 0.1$, we have $y(0) = c(0)V(0) = 0.1 \times 500 = 50$

Plug $t = 0$ into $(*)$, we get: $50 = \dfrac{k}{500^2}$. Thus, $k = 625 \times 10^5$

$$y(t) = \frac{625 \times 10^5}{(t+500)^2}$$

$$c(t) = \frac{y(t)}{V(t)} = \frac{625 \times 10^5}{(t+500)^3}.$$

For $c(t) = \frac{1}{8} c(0)$, we need

$$\frac{625 \times 10^5}{(t+500)^3} = \frac{1}{8}\, 0.1$$

Thus, $(t+500)^3 = \dfrac{625 \times 10^5}{0.1/8} = \dfrac{8 \times 625 \times 10^5}{0.1}$

$$= 8 \times 625 \times 10^6$$

$$= 2^3 \times 5^3 \times 100^3$$

Then $t + 500 = 2 \times 5 \times 100 = 1000.$

We get $t = 500$ (minutes)