Final Exam: Some problems for review

The Final exam covers the material in Sections 6.5, 6.6, 7.1-7.9. It is a closed-book exam, taking place in the regular classroom from 11 AM to 1 PM. Graphing calculators are allowed. The table of Laplace transform on page 252 of the textbook will be provided. Some problems you should know how to solve:

- Solve a mechanical vibration with impulse forcing.
- Find the convolution of two functions.
- Apply convolution to find inverse Laplace transform.
- The connected tank problem.
- Sketch a direction field of an autonomous system of ODEs.
- Categorize the equilibrium state (at the origin) of the system $Y' = AY$.
- Solve a system of homogeneous / nonhomogeneous linear ODEs.

1. Circle the set(s) of vectors that are linearly dependent.

   a. $x^{(1)} = \begin{bmatrix} 1 \\ t+1 \end{bmatrix}$, $x^{(2)} = \begin{bmatrix} t \\ t(t+1) \end{bmatrix}$

   b. $x^{(1)} = \begin{bmatrix} 1 \\ t+1 \end{bmatrix}$, $x^{(2)} = \begin{bmatrix} 2 \\ 2(t+1) \end{bmatrix}$

   c. $x^{(1)} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$, $x^{(2)} = \begin{bmatrix} \sin(-t) \\ \cos(-t) \end{bmatrix}$

   d. $x^{(1)} = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$, $x^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $x^{(3)} = \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}$

2. Consider the system $Y' = \begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix} Y$. What kind of equilibrium state is the origin?

   a. Stable node

   b. Unstable node

   c. Stable saddle point

   d. Unstable saddle point

   e. Stable spiral point

   f. Unstable spiral point

   g. Stable center point

   h. Unstable center point

   i. Stable improper node

   j. Unstable improper node

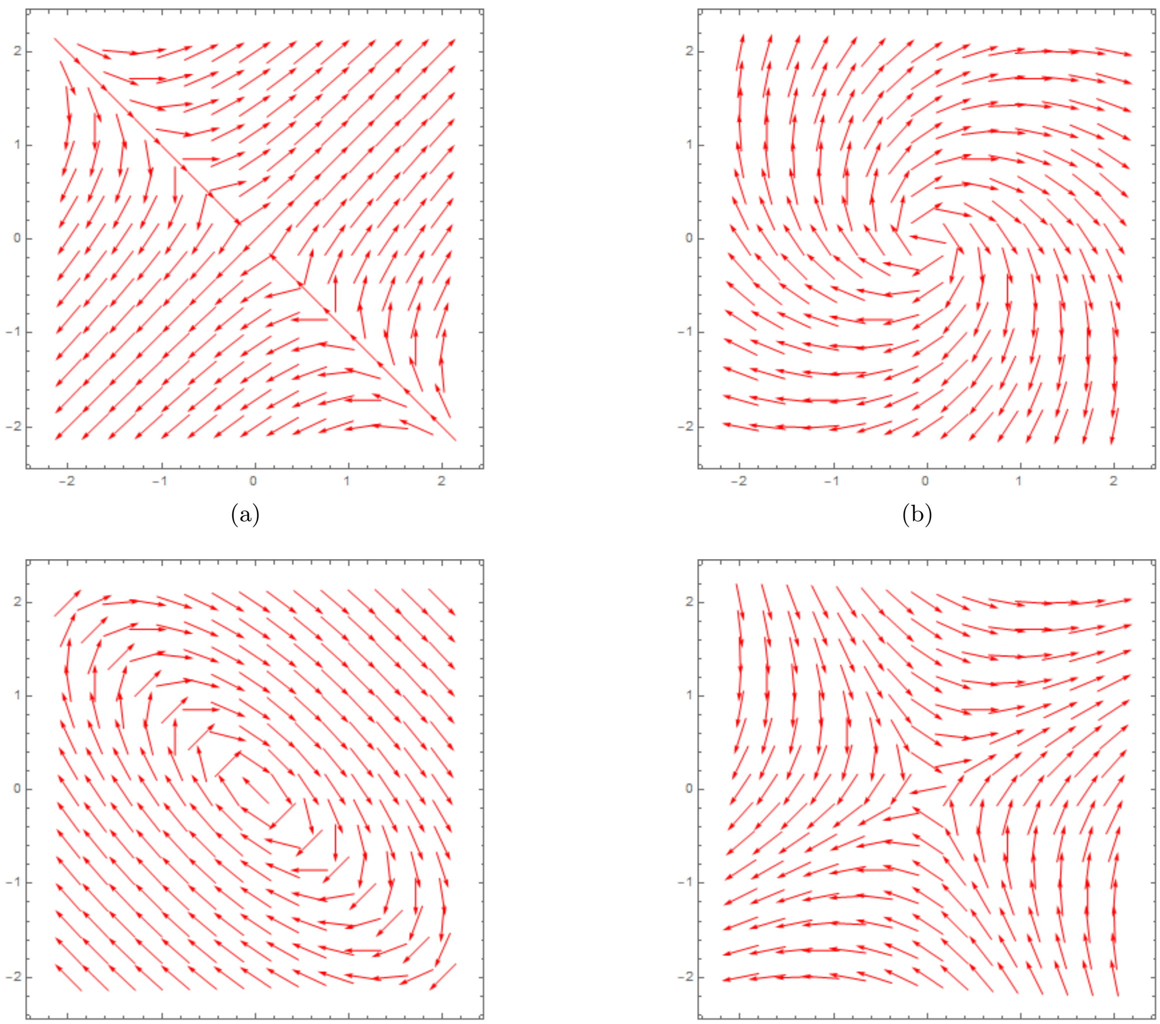
3. Which of the vector fields in Figure 1 is the direction field of the ODE $x' = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} x$?

4. Which of the vector fields in Figure 2 is the direction field of the ODE $x' = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} x$?

5. Sketch the direction field of the autonomous system

$$\begin{cases} x_1' = x_1^2 + x_2^2 \\ x_2' = 2x_2 + 1 \end{cases}$$

   at four points $(x_1, x_2) = (0,1), (0,0), (1,0), (1,1)$.



(a) (b) (c) (d)

Figure 1: Direction fields

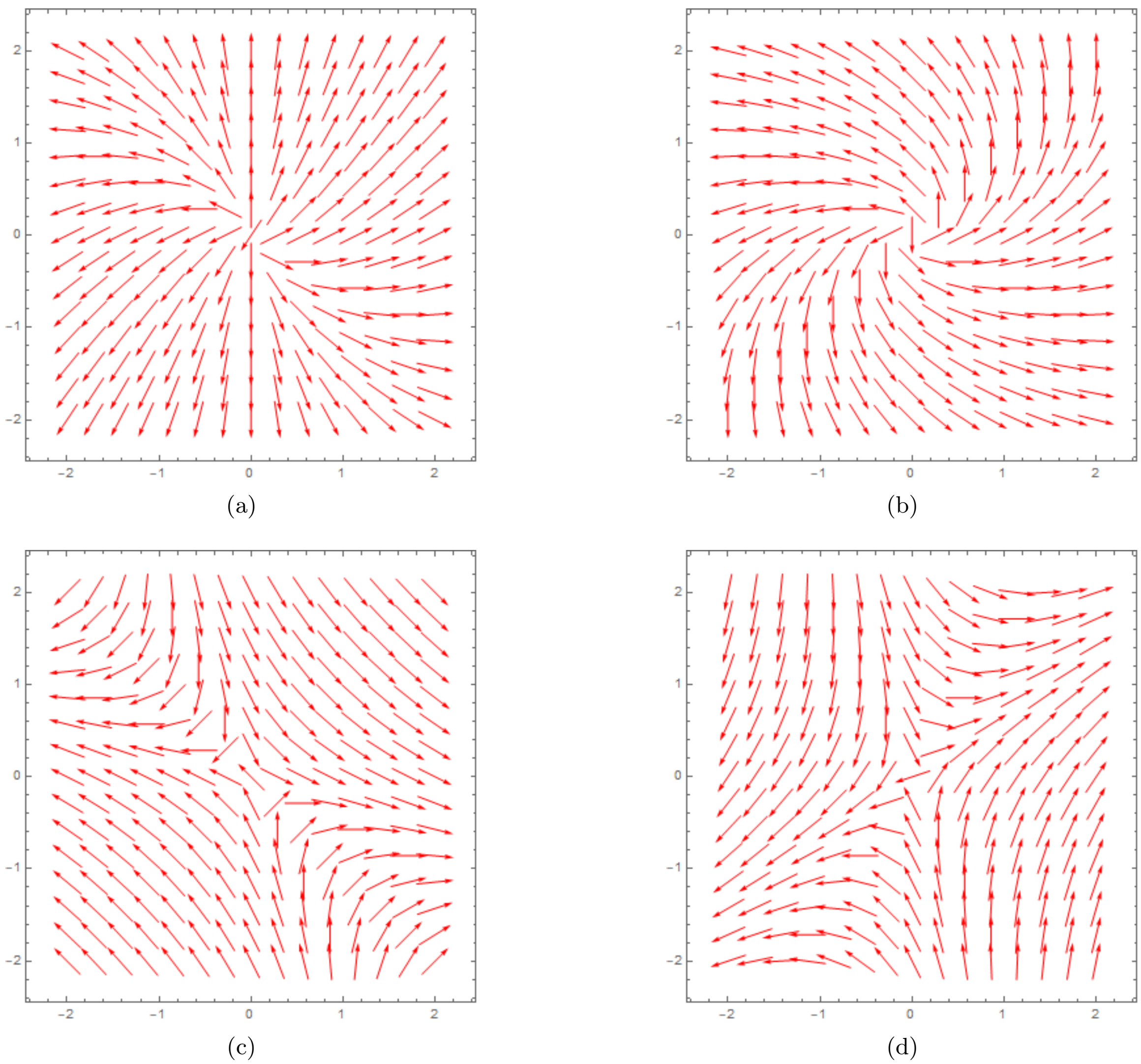


Figure 2: Direction fields

6. Consider a mass-spring system has $m = 1$, $\gamma = 2$, $k = 5$. The mass is initially pulled 2 (unit length) down below the equilibrium position and then released. An unit impulse force is applied upward to the mass at time $t = 1$. Write the equation of motion together with the initial conditions.

7. Consider two interconnected tanks. Tank 1 initially contains 30 gal of water and 2 lbs of salt. Tank 2 initially contains 20 gal of water and 1 lbs of salt. Fresh water is pumped into tank 1 at a rate of 1 gal/min. The mixture in tank 1 is pumped into tank 2 at a rate of 2 gal/min. The mixture in tank 2 is pumped back to tank 1 at a rate of 1 gal/min. At the same time, tank 2 is drained a rate of 1 gal/min. Write a system of ODEs that describes the amount of salt at each tank together with the initial conditions.

8. Let $x^{(1)} = \begin{bmatrix} t \\ e^t \end{bmatrix}$ and $x^{(2)} = \begin{bmatrix} t^2 \\ \cos t \end{bmatrix}$. Find the Wronskian of $x^{(1)}$ and $x^{(2)}$. Can they both solve an ODE of the form $x' = P(t)x$?

9. Let $v(t) = \begin{bmatrix} t+i \\ 2+it \end{bmatrix}$ and $a(t) = e^{2t+3i}$. Write $a(t)v(t)$ in complex standard form $v_1(t) + iv_2(t)$.

10. Is the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ diagonalizable? If write A in the form PDP^{-1} where P is an invertible matrix and D is a diagonal matrix.

11. Find the inverse Laplace transform of $F(s) = \frac{1}{(s+1)^2(s^2+1)}$.
Hint: use convolution or partial fractional decomposition.

12. Solve the initial value problem

$$\begin{cases} x_1' &= 3x_1 + 4x_2 + t \\ x_2' &= 4x_1 - 3x_2 + t^2 \end{cases}, \quad x_1(0) = 1, \quad x_2(0) = -1.$$

13. Let $A(t) = \begin{bmatrix} t^2 & t \\ t-1 & t+1 \end{bmatrix}$. Determine $A'(1)$ and $\int_0^1 A(t)^2 dt$.

14. Consider the initial value problem

$$\begin{cases} x_1' &= x_1 + x_2 \\ x_2' &= x_1 + x_3 \\ x_3' &= x_2 + 2x_3 \end{cases}, \quad x_1(0) = 0, \; x_2(0) = -1, \; x_3(0) = 1.$$

Write a third order ODE satisfied by x_2 together with initial conditions.

Answer keys:

1. a
2. f
3. a
4. b
5. Place vector $(1,3)$ at point $(1,0)$,...
6. $y'' + 2y' + 5y = -\delta(t-1)$, $y(0) = 2$, $y'(0) = 0$.
7. $$\begin{cases} y_1' &= -\frac{y_1}{15} + \frac{y_2}{20} \\ y_2' &= \frac{y_1}{15} - \frac{y_2}{10} \end{cases}, \quad y_1(0) = 2,\ y_2(0) = 1.$$
8. $t\cos t - t^2 e^t$. No
9. $v_1(t) = \begin{bmatrix} e^{2t} t\cos(3) - e^{2t}\sin(3) \\ 2e^{2t}\cos(3) - e^{2t} t\sin(3) \end{bmatrix}$ and $v_2(t) = \begin{bmatrix} e^{2t}\cos(3) + e^{2t} t\sin(3) \\ e^{2t} t\cos(3) + 2e^{2t}\sin(3) \end{bmatrix}$
10. Yes. $$A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$
11. $\frac{1}{2} e^{-t}\left(t - e^t\cos(t) + 1\right)$
12. $x_1(t) = \frac{1}{625}\left(384e^{-5t} + 274e^{5t} - 33 - 75t - 100t^2\right)$
    $x_2(t) = \frac{1}{625}\left(-768e^{-5t} + 137e^{5t} + 6 - 150t + 75t^2\right)$
13. $$A'(1) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \int_0^1 A(t)^2 dt = \begin{bmatrix} 1/30 & 13/12 \\ -3/4 & 13/6 \end{bmatrix}$$
14. $x''' - 3x'' + 3x = 0$, $x(0) = -1$, $x'(0) = 1$, $x''(0) = 0$.