Finding fundamental matrix

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$$Y' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} Y$$

Find a fundamental matrix $\phi(t)$ such that $\phi(0) = I_2$.

Recall:

① A fundamental matrix is a matrix $\phi(t) = \begin{bmatrix} | & | \\ y^{(1)} & y^{(2)} \\ | & | \end{bmatrix}$

where $\{y^{(1)}, y^{(2)}\}$ is a fundamental set of solutions.

② There are many fundamental matrices depending on how you choose the fundamental set of solutions.

First, you'll find one fundamental set of solutions.

$\lambda_1 = i, \quad \lambda_2 = -i, \qquad v_1 = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

$$y^{(1)} = v_1 e^{\lambda_1 t} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix} e^{it} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix} (\cos t + i \sin t)$$

$$= \begin{bmatrix} 2\cos t + \sin t - i\cos t + 2i\sin t \\ \cos t + i \sin t \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2\cos t + \sin t \\ \cos t \end{bmatrix}}_{u} + i \underbrace{\begin{bmatrix} -\cos t + 2\sin t \\ \sin t \end{bmatrix}}_{v}$$

One fundamental set of sols. is $\{u, v\}$.

One fundamental matrix is $\Phi_1(t) = \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} = \begin{bmatrix} 2\cos t + \sin t & -\cos t + 2\sin t \\ \cos t & \sin t \end{bmatrix}$

Unfortunately, $\Phi_1(0) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \neq I_2$

We need to "adjust" this fundamental matrix so that $\Phi(0) = I_2$.

Let chose a different fundamental set of sols:

$$\{c_1 u + c_2 v,\ d_1 u + d_2 v\}$$

these constants are to be determined

$$\Phi(t) = \begin{bmatrix} | & | \\ c_1 u + c_2 v & d_1 u + d_2 v \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}$$

$$\underbrace{\Phi(0)}_{I_2} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} \longrightarrow \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

Therefore,

$$\Phi(t) = \begin{bmatrix} | & | \\ 0u + v & u + 2v \\ | & | \end{bmatrix} = \underbrace{\begin{bmatrix} \cos t - 2\sin t & 5\sin t \\ -\sin t & \cos t + 2\sin t \end{bmatrix}}_{\text{final answer!}}$$