Lecture 10 (9/22/2021)

* Prayer
* Spiritual thought ....

* Linear 2^{nd} order ODE are of the form $a(t)y'' + b(t)y' + c(t)y = d(t)$

  One can write this ODE in a standard form:

$$y'' + p(t)y' + q(t)y = g(t)$$

We need two initial conditions to get a unique solution. In many physical problems, these conditions are $y(t_0) = y_0$, $y'(t_0) = y_0'$.

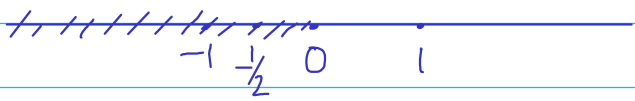
values of y and y' at time t_0

Unlike linear ODE of 1^{st} order, there is no general method to solve every linear ODE of 2^{nd} order. But we have abstract result about the existence and uniqueness:

Theorem: If $p(t), q(t), g(t)$ are continuous on an interval I containing t_0 then the IVP has a unique solution on I.

Ex $(t^2-1)y'' + \sqrt{t}\, y' + \ln(2t+1) y = 1$, $y(0) = 1$, $y'(0) = 2$

$$\rightsquigarrow\ y'' + \underbrace{\frac{\sqrt{t}}{t^2-1}}_{p(t)} y' + \underbrace{\frac{\ln(2t+1)}{t^2-1}}_{q(t)} y = \underbrace{\frac{1}{t^2-1}}_{g(t)}$$



A solution is guaranteed to exist and be unique on the interval $I = [0, 1)$.

An interesting feature about homogeneous linear ODEs is the superposition principle:

- One can add two solutions to get another solution
- One can scale a solution to get another solution

$$y'' + p(t)y' + q(t)y = 0$$

If y_1 and y_2 are solutions then $c_1y_1 + c_2y_2$ are also solutions for any constants c_1 and c_2.

Ex:

$$y'' + 3y' + 2y = 0 \quad \begin{cases} y_1 = e^{-t} \\ y_2 = e^{-2t} \end{cases}$$

Then $2e^{-t} - e^{-2t}$, $\pi e^{-t} - e^{1-2t}$, ... are also solutions.

This principle is, in general, not true for nonlinear ODEs.

Ex: $y' = y^2$ has a solution $y(t) = \dfrac{1}{1-t}$

But $\dfrac{2}{1-t}$ is not a solution!

How to solve ODEs of the form $ay'' + by' + cy = 0$?

Test this candidate: $y = e^{rt}$, where r is a constant to be chosen.

$$\underbrace{(ar^2 + br + c)}_{\text{choose } r \text{ that makes this equal } 0} e^{rt} = 0$$

Ex: $y'' + y' - 2y = 0$, $y(0) = 1$, $y'(0) = 2$

Characteristic eq: $r^2 + r - 2 = 0 \rightsquigarrow r = 1, r = -2$

Get two solutions: $y_1 = e^{t}$, $y_2 = e^{-2t}$

Solutions: $y = c_1e^{t} + c_2e^{-2t}$

Use the initial conditions to determine c_1 and c_2:

$$\left.\begin{aligned} 1 &= y(0) = c_1 + c_2 \\ 2 &= y'(0) = c_1 - 2c_2 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 4/3 \\ c_2 &= -1/3 \end{aligned}$$