Lecture 11

Thursday, September 23, 2021 10:24 PM

* Prayer
* Spiritual thought ....

Linear 2^{nd} order homogeneous ODE:

$$y'' + p(t)y' + q(t)y = 0 \qquad (*)$$

If y_1, y_2 are sols. then any linear combination of y_1, y_2 is also a solution: $y = c_1 y_1 + c_2 y_2$.

Use the initial conditions to determine c_1 and c_2.

Ex: With initial conditions $y(0) = 1$, $y'(0) = 2$, we get

$$\begin{cases} 1 = c_1 y_1(0) + c_2 y_2(0) \\ 2 = c_1 y_1'(0) + c_2 y_2'(0) \end{cases} \rightsquigarrow \text{get } c_1, c_2 \text{ by solving this system}$$

Can the system have no solutions? For example, can the system be like this

$$\begin{cases} 1 = c_1 + c_2 \\ 2 = 3c_1 + 3c_2 \end{cases} \quad ?$$

To answer this question, one needs to check if the determinant

$$\begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix}$$ is nonzero.

Let $W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

$\downarrow$

sometimes written as $W[y_1, y_2](t)$ or simply $W[y_1, y_2]$

Note:

$$\begin{aligned} &\left(y_1'' + p(t) y_1' + q(t) y_1 = 0\right) \times y_2 \\ - &\left(y_2'' + p(t) y_2' + q(t) y_2 = 0\right) \times y_1 \\ \hline &\underbrace{\left(y_1'' y_2 - y_2'' y_1\right)}_{W'} + p(t) \underbrace{\left(y_1' y_2 - y_2' y_1\right)}_{W} = 0 \end{aligned}$$

$\rightsquigarrow \quad W' + pW = 0$

$\rightsquigarrow \quad \boxed{W = C e^{-\int p \, dt}}$ (Abel's thm)

What we can learn: $W(t)$ is either 0 for all t or $\neq 0$ for all t.

If $W \neq 0$ then $\{y_1, y_2\}$ is called a <u>fundamental set of solutions</u>. This is because any sol. y to $(*)$ is a linear combination of y_1 and y_2.

optional

* From algebraic perspective:

- If y_1 and y_2 are linearly dependent (i.e. $y_1 = c y_2$ or $y_2 = c y_1$ for some constant c) then $W = 0$ for all t.
- If $W \neq 0$ for some t then y_1 and y_2 are linearly independent.

optional $\begin{cases} W \not\equiv 0 \text{ iff } y_1 \text{ and } y_2 \text{ are linearly independent. (Supposing they} \\ \text{solve } (*)) \end{cases}$

* Linear 2^{nd} order homogeneous ODE with constant coefficients:

$$ay'' + by' + cy = 0$$

Tool: test candidate of the form $y = e^{rt}$.

r has to solve the equation $\underbrace{ar^2 + br + c = 0}_{\text{Characteristic equation of the ODE}}$.

Ex: $y'' + y' - 2y = 0 \rightsquigarrow r^2 + r - 2 = 0 \rightsquigarrow r = 1, -2$ $\begin{cases} y_1 = e^t \\ y_2 = e^{-2t} \end{cases}$

$\underset{(**)}{y'' + 2y' + 3y = 0} \rightsquigarrow r^2 + 2r + 3 = 0 \rightsquigarrow r = -1 \pm i\sqrt{2}$

$y_1 = e^{(-1+i\sqrt{2})t}$, $y_2 = e^{(-1-i\sqrt{2})t}$

Euler's formula:

$$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b) = e^a \cos b + i e^a \sin b.$$

Thus, $y_1 = e^{-t}\cos(\sqrt{2}t) + ie^{-t}\sin(\sqrt{2}t)$

$y_2 = e^{-t}\cos(\sqrt{2}t) - ie^{-t}\sin(\sqrt{2}t)$

$y_3 = \frac{y_1 + y_2}{2} = e^{-t}\cos(\sqrt{2}t)$ is a solution to $(**)$.

$y_4 = \frac{y_1 - y_2}{2i} = e^{-t}\cos(\sqrt{2}t)$ is a solution to $(**)$.