Lecture 12 (9/27/2021)

* Prayer
* Spiritual thought...

* Solving for linear 2^{nd} order homogeneous ODE with constant coefficients:

$$ay'' + by' + cy = 0 \qquad (*)$$

Method: test candidate of the form $y = e^{rt}$

$\rightsquigarrow$ r must be a root of $\underbrace{ar^2 + br + c = 0}_{\text{characteristic equation}}$.

$\rightsquigarrow$ root r_1, r_2

$\rightsquigarrow$ $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$

If $r_1 \neq r_2$ then any sols to $(*)$ is of the form $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

This method can be applied for ODEs of first order.

Ex:

$$y' + 2y = 0 \qquad (**)$$

$\rightsquigarrow$ $y = e^{rt}$ $\rightsquigarrow$ $(r+1)e^{rt} = 0$ $\rightsquigarrow$ $r = -1$ $\rightsquigarrow$ $y = e^{-t}$

Any sol to $(**)$ is of the form $y = Ce^{-t}$.

The case of complex roots

$$y'' + 2y' + 3y = 0$$

Characteristic eq: $r^2 + 2r + 3 = 0$ $\rightsquigarrow$ $r = -1 \pm i\sqrt{2}$.

$y_1 = e^{(-1+i\sqrt{2})t}$, $y_2 = e^{(-1-i\sqrt{2})t}$.

these solutions don't look very nice because they are complex.

Fix:

$$y_3 = \frac{y_1 + y_2}{2} = e^{-t}\cos(\sqrt{2}t)$$

$$y_4 = \frac{y_1 - y_2}{2i} = e^{-t}\sin(\sqrt{2}t)$$

* Euler's formula: $e^{a+ib} = e^a e^{ib} = e^a(\cos b + i\sin b)$

Ex: $y'' + 2y' + 2y = 0$, $y(0) = 2$, $y'(0) = -1$.

The case of repeated root

$$y'' + 4y' + 4y = 0 \rightsquigarrow r^2 + 4r + 4 = 0 \rightsquigarrow r = -2 \text{ (double root)}$$

$\rightsquigarrow e^{-2t}$ is a sol.

We still need one more solution in order to form a fundamental set of sols.

To see this solution, we use the integration factor method:

$$(y'' + 4y' + 4y = 0) \times e^{2t}$$

$$\rightsquigarrow \quad (e^{2t} y)'' = 0$$

$$\rightsquigarrow \quad e^{2t} y = c_1 + c_2 t$$

$$\rightsquigarrow \quad y = c_1 e^{-2t} + \underbrace{c_2 t e^{-2t}}_{\text{this is the second solution}}$$

Conclusion: If the characteristic eq. $ar^2 + br + c = 0$ has a double root r_0 then $y_1 = e^{r_0 t}$ and $y_2 = t e^{r_0 t}$ form a fundamental set of sols.

* Another way to see the second solution: (variation of parameters)

$y = C e^{-2t}$ is a sol for any constant C

Now allow C to depend on t.

$$
\begin{array}{rl}
 & y'' = C'' e^{-2t} - 2C' e^{-2t} + 4C e^{-2t} - 2C' e^{-2t} \\
+ & (y' = C' e^{-2t} - C 2 e^{-2t}) \times 4 \\
 & (y = C e^{2t}) \times 4 \\
\hline
 & C'' e^{-2t} = 0 \rightsquigarrow C'' = 0 \rightsquigarrow C = c_1 t + c_2
\end{array}
$$

Thus, $y = (c_1 t + c_2) e^{-2t}$.