Lecture 13 (9/29/2021)

* Prayer
* Spiritual thought .....

* Summary of the methods to solve for ODE of the form $ay''+by'+cy=0$.
  - Characteristic equation: $ar^2+br+c=0$
  - Solve for roots:
    - → distinct real roots r_1, r_2 : $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
    - → complex roots $r = \alpha \pm i\beta$: $y = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$
    - → double root r : $y = c_1 e^{rt} + c_2 t e^{rt} = e^{rt}(c_1 + c_2 t)$

Ex: Write the general solution to each ODE

(a) $2y'' + 3y' + y = 0$

(b) $y'' + 4y' + 5y = 0$

(c) $4y'' + 4y' + y = 0$

* Reduction of order:

If we know a root to a polynomial of degree d, we can factor that polynomial. The task of finding the other roots boils down to finding roots of a polynomial of degree $d-1$.

Ex:

$$x^3 + x^2 + x + 1 = 0 \quad , \quad x = -1 \text{ is a root}$$

$$\hookrightarrow (x+1)(x^2+1) = 0$$

At the level of ODE, the situation is similar, although more computations are required.

Ex: $y'' + ty' - 2y = 0$

One solution is $y = t^2 + 1$. How to find one more solution?

Any function of the form $y = C(t^2+1)$ is also a solution.

Now allow C to depend on t: $\quad y = v(t)\,(t^2+1)$.

$$(-2)\times\left(y = (t^2+1)v\right)$$
$$t\times\left(y' = 2tv + (t^2+1)v'\right)$$
$$y'' = 2v + 4tv' + (t^2+1)v''$$
$$0 = (t^2+1)v'' + (t^3+t+4t)v'$$

$$\rightsquigarrow \quad v'' + \frac{t^3+5t}{t^2+1}v' = 0$$

Put $u = v'$. Then solve for u using the integrating factor.

* Non-homogeneous linear 2^{nd} order ODE:

$$y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$

Note: if y_1 and y_2 are sol. to $(*)$ then the difference $y_1 - y_2$ solves to homogeneous equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (**)$$

A general solution to $(*)$ is $y = \underbrace{y_p}_{\text{particular sol. to } (*)} + \underbrace{y_c}_{\text{general sol. to } (**)}$

How to determine a particular solution?

Ex

$$y'' - 3y' + 2y = e^{-t}$$

Candidate: $y = Ae^{-t}$

Substitute into the ODE: $\quad Ae^{-t} + 3Ae^{-t} + 2Ae^{-t} = e^{-t} \rightsquigarrow 6A = 1 \rightarrow A = 1/6$

This method is known as the method of undetermined coefficients.

* Step by step solution to the example

$$y'' + ty' - 2y = 0 \qquad (*)$$

knowing one solution $y_1 = t^2+1$.

Write $y = (t^2+1)v$ where v is a function of t to be determined.
Substitute this function into $(*)$ and reduce:

$$(t^2+1)v'' + (t^3+5t)v' = 0$$

Put $u = v'$. Then

$$(t^2+1)u' + (t^3+5t)u = 0$$

$$\rightsquigarrow \quad u' + \underbrace{\frac{t^3+5t}{t^2+1}}_{p(t)} u = 0 \qquad (**)$$

One can find u by the integrating factor method:

$$\int p(t)\,dt = \int \left(t + \frac{4t}{t^2+1}\right)dt = \frac{t^2}{2} + 2\ln(t^2+1).$$

Integrating factor: $\mu(t) = \exp\left(\frac{t^2}{2} + 2\ln(t^2+1)\right) = e^{\frac{t^2}{2}} e^{2\ln(t^2+1)}$

$$= e^{\frac{t^2}{2}}(t^2+1)^2$$

Multiplying both sides of $(**)$ by $\mu(t)$, we get

$$(\mu u)' = 0 \quad \rightsquigarrow \quad u = \frac{C}{\mu} = C e^{-t^2/2}(t^2+1)^{-2}.$$

To get v, we integrate u: $\quad v = C\int e^{-\frac{t^2}{2}}(t^2+1)^{-2}\,dt$

$$= C_1 \int_0^t e^{-\frac{s^2}{2}}(s^2+1)^{-2}\,ds + C_2$$

Therefore, all solutions to $(*)$ are

$$y = (t^2+1)v = C_1(t^2+1)\int_0^t e^{-s^2/2}(s^2+1)^{-2}\,ds + C_2(t^2+1)$$